PHILOSOPHICAL

TRANSACTIONS

OF THE

ROYAL SOCIETY

OF

LONDON.

FOR THE YEAR MDCCCXXX.

PART II.

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MDCCCXXX.

ADVERTISEMENT.

Since the appearance of the Papers relating to the Paramatta Observations, which were published at the request and at the expense of the Colonial Office, and appended to the Philosophical Transactions, it has appeared that some of the Observations communicated by Mr. Rumker were made by that gentleman while he was the paid assistant of Sir Thomas Brisbane, at an Observatory founded by Sir Thomas Brisbane, and with his instruments; and that some others were actually made by Sir Thomas Brisbane himself.

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Meteorological Journal, kept at the Apartments of the Royal Society, by order of the President and Council.

ERRATA in Part I.

Page 74, *dele* the last paragraph but one.
—— 142, in the second formula, line 4th, *for* ϕ, *read* 1.

Part II.

Page 260, last line, *for* fire *read* ice.
—— 282, line 8, *for* heated *read* luted.

A List of Public Institutions and Individuals, entitled to receive a copy of the Philosophical Transactions of each year, on making application for the same directly or through their respective agents, within five years of the date of publication.

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Professor Bessel.
Dr. William Olbers.

PHILOSOPHICAL TRANSACTIONS.

XVII. *Memoir on the occurrence of Iodine and Bromine in certain Mineral Waters of South Britain.* By CHARLES DAUBENY, *M.D. F.R.S. Professor of Chemistry in the University of Oxford.*

Read May 6, 1830.

THE discovery in sea-water of iodine and bromine, two principles which, although in minute proportions, are said to be generally diffused throughout the present ocean, naturally suggested the inquiry, as to whether these same ingredients might not be found to exist in springs occurring in inland situations when containing a similar saline impregnation. This accordingly has been already determined by STROMEYER, LIEBIG, and others, to be the case in many of the brine-springs of Germany, France, and Italy; but at the time my attention was first directed to the subject, I was unacquainted with any trials of the kind having been instituted with reference to those of this country, except by Professor TURNER of the London University, regarding the presence of iodine in the mineral waters of Scotland; in only one of which, that of Bonnington near Leith, he appears to have detected it. I was therefore induced in the course of last spring and summer to undertake a pretty extensive survey of such English springs as are known to contain a considerable proportion of common salt; and having succeeded in detecting in several of them traces of one or both the substances alluded to, I inserted a brief account of the results obtained, in the Philosophical Magazine and Annals of Philosophy for September last.

An article that has appeared in a subsequent number of the same periodical work has, however, been the means of drawing my attention to a little work

by Mr. Murray, entitled "Experiments on Chemical Philosophy," which had not before fallen in my way; and from this it is clear, that the detection of iodine in the Gloucester Spa water had been made by that gentleman some time before I had engaged in the inquiry. I am unable, however, to discover in his publication, although it bears so late a date as 1828, any thing that can substantiate the assertion which its author has made in the number of the Philosophical Magazine and Annals referred to, as to his having anticipated me in the discovery of iodine in the springs at Cheltenham*, or in that of bromine in those of Ingestrie. I consider myself, therefore, still warranted in claiming as my own the first public announcement of the existence of bromine in our English springs; but I am far from attaching importance to a discovery which had been previously made in so many similar situations abroad, and would wish it to be understood, that my only pretence for offering to the Royal Society the present communication, is the circumstance of my having examined on the spot most of the mineral springs hereafter enumerated, and having undertaken, wherever it appeared practicable, to obtain an approximation at least to the proportion which these principles bore to the other ingredients present, and to estimate their comparative frequency and abundance in the several rock-formations.

To the geologist, the results of such an inquiry may be of interest, as tending to identify the products of the ancient seas in their most minute particulars with those of the present ocean: and to the physician it may be an object of curiosity, to speculate how far the unexplained virtues attributed to certain mineral waters depend on the presence of these ingredients, the energy of whose action may perhaps compensate for the minute quantity in which they are found. I confess, indeed, that with regard to the former of them, Iodine, we ought to be sceptical as to any medicinal agency that can be exerted by so small a quantity as a single grain diffused through ten gallons of water, the largest proportion in which I have ever detected it. But with respect to the second, Bromine, after considering the statements of its discoverer M. Balard, as to its highly poisonous operation upon animals, which my own experience

* Mr. Ainsworth, however, one of the editors of the Edinburgh Journal of Natural and Geographical Science, states, that he had communicated the fact of the existence of iodine in the Cheltenham waters previously to my announcement of it.

of the irritating effects of its vapour tends fully to confirm, I cannot view it as absurd to trace the medical virtues ascribed to such waters as those of Ashby-de-la-Zouch, to the presence of even so small a quantity as a grain of hydro-bromate of magnesia (if such be the combination) in each pint of the water; and that the proportion would not fall far short of that, my experiments on this particular spring seem to warrant me in concluding. It is curious at least, that almost the only two brine-springs, properly so called, which have acquired any reputation as medicinal agents, that of Kreutznach in the Palatinate, and that of Ashby-de-la-Zouch in Leicestershire, both should contain a larger proportion than common of this new principle; and that in either instance that reputation should have been enjoyed, long before any suspicion as to their peculiar nature could have been entertained.

The objects I had in view in this inquiry being what are above stated, I have chosen to classify the springs noticed in the accompanying Table according to the geological position of the strata from which they issue; and under the head of each have set down the total amount of their saline ingredients; the nature and proportion of them as ascertained by former chemists, or, whenever I could not depend upon the results, by myself; and the proportion which the iodine and bromine, where either of these principles existed, bore to the quantity of water, and likewise to that of the chlorine which the solid ingredients of the spring might contain. The latter statement has been introduced in order to remove an impression which may have been created in consequence of the detection of iodine, as it is said, even in common pump-water*, when very large quantities of it were evaporated; from which circumstance it might be inferred, that this principle is not only a constant accompaniment of common salt, but that its quantity bears a pretty regular ratio to that of the latter ingredient. Although I have myself evaporated no less than forty-eight gallons of the Oxford pump-water without finding the slightest trace of iodine in the last portions, I shall not dispute the truth of the former position, which might possibly have been borne out, had still larger quantities been operated upon†; but that the latter opinion is untenable, will be readily seen from the

* Mr. Cuff, a chemist at Bath, has also detected it in the hot springs of that place, by evaporating about thirty gallons of the water.

† I am also loth to question the fact (stated on good authority) of the existence of a minute pro-

accompanying Table, which shows that the proportion of iodine to chlorine varies in every possible degree, and that the springs most strongly impregnated with common salt are in some instances those in which I have evaporated the largest quantity without detecting any trace of iodine. The same remark will equally apply to bromine; so that the general inference seems to be, that although these two principles may perhaps be never entirely absent where the muriates occur, yet that their distribution is certainly very unequal, and therefore forms a proper subject of scientific research. The quantity in which the former of these ingredients occurs in mineral waters is commonly so inconsiderable, that I have been unable to determine it by direct analysis, and have been therefore obliged to content myself with obtaining an approximation to its real amount. In the case of one of the Leamington springs, indeed, I employed the agency of nitrate of silver to precipitate the iodine from the concentrated water, and afterwards separated by means of ammonia the chloride from the iodide of silver obtained. I have reason, however, to believe, from some comparative experiments, that where the proportion which this latter ingredient bears to the former one is extremely small, it may be taken up either wholly or in part by ammonia; and I therefore contented myself in other instances with evaporating the water until it began to produce the characteristic blue or violet tinge with starch and sulphuric acid. This was then compared with the colour imparted by the same test to a solution of hydriodate of potass of known strength; and the latter, if not of the same shade already, was brought to it by dilution with a measured quantity of water. Having thus noted the proportion of iodine in the test liquor with which the concentrated solution corresponded, it was easy to calculate what it must have been in the mineral water itself, by knowing the number of times its original quantity had been reduced by evaporation previously to the employment of this re-agent.

The sulphates and muriates present in brine-springs do not appear to interfere with the delicacy of this test; but where bromine was also present, I have

portion of iodine in sea-water, although I have reduced ten gallons of it, taken from the English Channel near Cowes, to less than half an ounce, without being able to detect any in the residuum. There seems reason, however, to infer, from what is stated in the next page, that the starch test cannot be relied upon to detect very minute quantities of iodine, when a comparatively large proportion of bromine is present in the same solution.

seen the liquor, either at the time or shortly after the operation of the re-agent, assume a pinkish hue, owing, as I suppose, to the reddish tinge of the bromine given out mixing with the blue colour of the iodide of starch. In stating therefore, as I have done in the Table, the proportions of iodine in several of the waters, I am far from pretending to offer more than an approximation to the relative quantity in which it occurs, and am fully aware of the necessity of more precise experiments, conducted on a different principle, before the points in question can be considered as satisfactorily determined.

The starch test I find will readily indicate a quantity of iodine not exceeding one grain to 7 gallons of water, or one 450,000dth part; but as in no case that has occurred to me the proportion exceeded one grain to 10 or 12 gallons, and in many appeared scarcely to amount to one-10th of that quantity, I despaired of arriving at more accurate results, by adopting any other method that aspired to greater precision than the one already stated.

In every case in which I have noted that no iodine could be detected, the water had been concentrated at least as far as to one-30th of its original quantity without effect; so that the proportion of this principle, supposing after all any of it to exist, could not well amount to a grain in 200 gallons. In some cases, indeed, where the spring was one of weak impregnation, I have carried the concentration much further, as may be seen in the Llandrindod waters, where no traces of iodine appeared, until they had been reduced to nearly one-50th of their original volume.

In my trials for bromine, I have in great measure conformed to the directions of Balard; first boiling down the water to about a fourth of its original quantity with a portion of quicklime to prevent the bromine from being dissipated by the heat; and then, after filtering the residuum, introducing chlorine as long as any sensible yellowness was caused by its addition. The water was then strongly agitated with ether, which collects on the surface, carrying with it the bromine with which it had combined, and was then poured off into a separate vessel. [The bromine, immediately upon being thus removed from the water, was treated with a quantity of a concentrated solution of pure soda sufficient to render the ether containing it colourless; the latter alkali being employed for this purpose in preference to the vegetable one, as I found that bromine formed with sodium a salt more soluble in alcohol than it did with potassium.]

Unfortunately, however, the salts which are contained in or deposited from the ethereal solution after the addition of the soda, appear to be of a very mixed description, consisting not only of the hydrobromate and bromate of soda, but also of the muriate and chlorate, together with a little uncombined alkali, if the proportion of the sodium to the bromine is not very nicely adjusted. I therefore began by heating the whole product sufficiently to convert the bromate of soda into the bromide, and the chlorate into the chloride, of sodium; and afterwards, in order to ensure the union of any alkali which may have been in excess with carbonic acid, I dissolved the whole in water impregnated with that gas. The solution was then brought to dryness, and strong alcohol added to separate the bromide of sodium as much as possible from the other ingredients; after which, the alcoholic solution, having been evaporated, was re-dissolved in water, and nitrate of silver added to it in order to form the insoluble bromide of silver, the weight of which, when dried and melted, would determine that of the bromine present, every 100 grains according to M. Balard indicating 41.1 of this principle.

From the weight of the precipitate, however, I felt myself obliged to make a large deduction, in proportion to the quantity of alcohol employed, for the chloride of sodium at the same time taken up; having ascertained by a previous experiment how much common salt a given quantity of this menstruum could dissolve. The latter part of the process, however, being liable to some uncertainty, I should have preferred, had my engagements permitted, re-examining the waters on the spot, and operating on such quantities of them as would have enabled me to extract appreciable quantities of bromine. This indeed I have done in the case of the Middlewich water, but not with sufficient attention to the quantities employed and obtained, to enable me to calculate in this manner the exact proportions between them: with regard to the other springs, the quantity of water which I could conveniently transport to my laboratory was not such as to enable me to pursue with much hope of success this particular method. It is therefore with diffidence that I offer provisionally the statements given in the Table, as an approximation to the relative quantities of bromine existing in some of our English springs, calculated according to the scheme of analysis above stated; and shall hope at some future and not very distant period to obtain results more worthy of reliance, should my

further labours on this subject not be rendered in the mean time unnecessary by the investigations of some other chemist. In cases where the quantity of this principle appeared to be less considerable, as in the Leamington, Cheltenham, and Gloucester waters, I have contented myself with guessing at its proportion by concentrating the water until it assumed a decidedly yellow tinge with chlorine, noting what proportion of bromine in water produced a colour of equal intensity.

The earliest of the rock-formations in this country that come under our consideration with reference to the present inquiry, is the greywacke slate of North Wales, which in the neighbourhood of Bualt in Radnorshire gives out springs containing a notable proportion of common salt. Those of Llandrindod have long enjoyed some reputation as medicinal agents, but their composition does not appear to have been correctly ascertained; for the most modern analysis I have seen* assigns to them a considerable proportion of muriate of magnesia, of which I find scarcely a trace. The more newly discovered springs at Bualt itself, though less celebrated, are similar in point of constitution; and being double the strength of those of Llandrindod, ought to possess superior medical virtues. At both places are waters which differ from the rest in containing an unimportant impregnation of sulphuretted hydrogen, but in other respects correspond.

Many of our coal-pits emit streams of salt water; but the most remarkable spring of the kind is that already noticed, of Ashby-de-la-Zouch in Leicestershire, which for the last few years, especially since the erection of the baths, which are now so great an ornament to the spot, has acquired a certain local reputation in the cure of diseases. Previously to the discovery of bromine, of which I detected in this water an appreciable quantity, Dr. Thomson of Glasgow had examined its composition; and I have therefore been satisfied with adhering to the results of his analysis, which is stated in the Table.

The most important, however, of the salt-springs that we meet with in this country are those in the new red sandstone formation of Cheshire; for an analysis of which I may refer to a paper of Dr. Henry's, published in the Transactions of this Society. In this instance, also, I have adopted the statements of another, merely making a proportionate deduction from the amount

* Analysis of the Llandrindod Waters; by Mr. Williams, Surgeon, 1819.

of the ingredients given by that able chemist, in consequence of the weaker impregnation of the samples of water I employed, than of those on which he appears to have operated.

It will be seen by reference to the Table, that all the brine-springs of that district contain bromine, and most of them iodine; indeed it is probable that if I had had time to concentrate larger quantities of the water, the latter would have been detected throughout. It may be remarked, however, that the rock-salt of Northwich in Cheshire contains no trace of either principle; a circumstance explicable from the more deliquescent nature of the hydriodic and hydrobromic salts, which would cause them, together with the earthy muriates, to remain in the mother-liquor after the common salt had crystallized, and thus to become distributed through the substance of the marly beds afterwards formed over the rock-salt, from which the brine-springs appear to derive their saline impregnation.

There is a blue variety of rock-salt met with at Ischel near Saltzburg, which, from the resemblance between its colouring matter and that of the compound of starch and iodine, might be suspected to contain this latter principle united with some kind of vegetable matter. I have been unable to obtain a specimen deeply enough tinged with the colouring matter alluded to, to set the question completely at rest; but on dissolving a portion of the blue salt, which I obtained through the kindness of Mr. Heuland, in water, not the slightest tinge appeared to be communicated to the solution, neither did any blueness appear on re-crystallizing the salt. The specimen alluded to gave no indications of iodine when tested with starch in the usual manner, and was nearly pure from admixture with foreign ingredients, although it appeared to contain a trace of sulphuric acid and of lime. At present, therefore, I am inclined to attribute the colour rather to some peculiar arrangement of the particles of the common salt itself, than to the presence of any other ingredient.

The springs containing purgative salts, which arise from the lias clay in various places along its whole range from Leamington to Gloucester, appear to be derived from the same source as the brine-springs of Cheshire and Worcestershire above alluded to; but their saline contents have been modified by the sulphuric acid generated by the decomposition of the sulphuret of iron present in the stratum from which they immediately proceed. Hence the pro-

portion of earthy muriates is usually greater in them than in the brine-springs properly so called; because the muriatic acid disengaged by the action of the sulphuric acid upon the common salt has dissolved a fresh quantity of lime or magnesia from the surrounding materials of the rock.

If such be the origin of the sulphates of soda and magnesia which impart to these waters their aperient quality, it would be natural to expect that they should be found in greater abundance on the first discovery of the spring than after it has been long drawn upon; and hence, perhaps, the remarkable discrepancy between the results of my examination of the Gloucester and some other waters, and those given on the authority of former chemists, may be explained, without impeaching the accuracy of either.

It will be seen by reference to the Table, that I have represented the ingredients of the Leamington waters on the authority of Dr. Thomson, as stated by Dr. Loudon in his Practical Dissertation* on these springs; and those of Cheltenham, with the exception of one lately discovered at Pittville, on that of Dr. Scudamore. The springs of Tewkesbury and Gloucester I have myself examined; there being of the former no analysis at all, and of the latter only one by Mr. Accum, which I had reason to believe, what I in fact found, quite inapplicable to its present composition. The spring which goes by the name of the Chalybeate Saline is at present destitute of iron, which I am assured it formerly possessed, whilst the Sulphureous contains no trace of sulphuretted hydrogen. These two springs, which at present appear almost identical, are the ones most strongly impregnated with purgative salts, and therefore approximate more nearly to the character of that analysed by Mr. Accum, according to his representation, than either of those termed "the pure saline," which he professed to have examined. Many of these springs it will be seen contain traces of bromine and iodine; but they seem to be less common in the aperient waters which are occasionally met with in the chalk and tertiary districts of this country; for I have examined three—those, namely, of Epsom, of Chad's Well in Gray's-Inn Lane, and of St. Leonard's Hill near Windsor, without discovering traces of iodine in any one. In the Epsom water alone a slight trace of bromine was perceptible.

With regard to the state of combination in which these principles occur, I

* Leamington, 1828.

have only to observe, that they are no doubt combined with hydrogen, forming the hydriodic and hydrobromic acids, and neutralized in all probability by magnesia, both forming with this basis salts decomposable at a low temperature, which seems to be the case with the compounds of both bromine and iodine existing in the waters I have examined. Even long continued boiling, there is reason to believe, diminishes the quantity of bromine originally present; and hence it seems advisable, when the object is to estimate the whole of this principle which a mineral water may contain, to combine the hydrobromic acid with lime, in the manner which I have recommended to be done when speaking of the mode of separating bromine from its combination.

I may conclude by observing, that there is little question as to the possibility of procuring a sufficient supply of bromine from our English brine-springs, should a demand be created for this new substance, either for medical purposes or for the arts of life; for, from a few rough trials of its comparative abundance in the Middlewich and Ashby springs, and in those of Kreutznach in the Palatinate, which affords, it is said, the principal supply for present consumption, I should regard our own quite as highly charged: neither can it be doubted but that the proportion of bromine present in many brine-springs exceeds considerably that contained in the present ocean, which, from experiments recently made by myself on water taken from the English Channel a short distance from Cowes, I have stated in the Table as existing in the proportion of one grain to the gallon.

TABLE comprehending a List of certain Springs in South Britain, which contain Common Salt in considerable quantity.

TABLE comprehending a List of certain Springs in South Britain, which contain com- the other Ingredients

Geological Position.	Locality of the Spring.	Name of the Spring.	Total of its Saline Contents in the Pint.	Iodine. Its proportion to The Water.	Iodine. Its proportion to The Chlorine.
TRANSITION SLATES.	Llandrindod, Radnorshire.	No. 1. The Pure Saline.	31.35 Gr.	Seems not to exceed 1 Grain to 343 Gallons.	As 1 to 50.000
	Ditto	No. 2. The Chalybeate Saline.	Nearly as No. 1.	None detected.	———
	Bualt, Radnorshire.	No. 1. The Saline.	77.6 Gr.	A trace, nearly the same as that exhibited by No. 1. Llandrindod.	As 1 to 120.000
	Ditto	No. 2. Sulphuretted Saline.	Nearly as No. 1.	None detected.	———
	Ditto	No. 3. The Chalybeate Saline.	Nearly as No. 1.	None detected.	———
COAL FORMATION.	Ashby-de-la-Zouch, Leicestershire.	The Moira Brine Spring.	179.88 Gr.	None detected.	———
	Near Newcastle-upon-Tyne.	The Walker Colliery Brine Spring.	192.0 Gr.	None detected.	———
	Kingswood, near Bristol.	The Soundwell Colliery Brine Spring.	64.0 Gr.	None detected.	———
	Northwich, Cheshire.	Brine Spring.	1696 Gr.	None detected.	———
	Middlewich, Cheshire.	Ditto	1824 Gr.	A trace, probably not exceeding 1 Grain to 343 Gallons.	As 1 to 2.650.000
	Nantwich, Cheshire.	Ditto	1760 Gr.	1 Grain to about 12 Gallons.	As 1 to 96.000
	Wheelock, Cheshire.	Ditto	1440 Gr.	A trace, apparently not greater than that in the Middlewich.	Nearly as 1 to 2.000.000
	Droitwich, Worcestershire.	Ditto	1746 Gr.	None detected.	———
	Shirleywich, Staffordshire.	Ditto	1552 Gr.	None detected.	———
LIAS CLAY.	Leamington, Warwickshire.	No. 1. Robbins's Well.	95.948 Gr.	1 Grain to about 10 Gallons.	As 1 to 3440
	Ditto	No. 2. Royal Pump Saline Spring.	134.749	Rather less than in No. 1.	———
	Ditto	No. 3. Smith's Pump.	109.992	A trace, but apparently not more than a Grain to 192 Gallons.	As 1 to about 48.000
	Ditto	No. 4. Wise's Pump.	107.396	Nearly as the preceding.	———

mon Salt in considerable quantity; together with a Statement of the Proportion of this and present in a Pint of each.

Bromine.		Chlorides of			Sulphates of			Per-oxide of Iron.	Carbonates or other Salts.	Authority on which the Statement of the Saline Ingredients is given.
Its proportion to The Water.	The Chlorine.	Calcium.	Magnesium.	Sodium.	Lime.	Magnesia.	Soda.			
Not estimated, but a distinct trace.	Not estimated.	6.6	24.75							DAUBENY.
As No. 1.	———	Ingredients as No. 1. except that it contains						0.6		DAUBENY.
A trace, not estimated.	———	11.2	A trace	66.4						DAUBENY.
A trace, not estimated.	———	Ingredients nearly as No. 1.								DAUBENY.
A trace, not estimated.	———	Ingredients as No. 1. except that it contains						A trace		DAUBENY.
1 Gallon seems to contain 4.68 Grains.	As 1 to 180	36.4	3.72	133.0	4.24		2.52			THOMSON.
A trace, not estimated.	———	2.8	2.8	186.0						DAUBENY.
None detected.	———	2.5		58.5			3.0			DAUBENY.
1 Gallon seems to contain 1.2 Grains.	As 1 to 6600	0.42	1.27	1667.0	25.5				Insoluble matter 1.696	HENRY.
9.36 Grains of Brome in 1 Gallon.	As 1 to 860	0.5	1.37	1793	27.35				Insoluble matter 1.3	HENRY.
6.32 Grains of Brome in 1 Gallon.	As 1 to 1275	0.45	1.32	1730	26.5				Insoluble matter 1.76	HENRY.
A trace, not estimated.	———	0.36	1.1	1415	22.0				Insoluble matter 1.44	HENRY.
None detected.	———		A trace	1691			40.25			DAUBENY.
4.32 Grains of Brome to 1 Gallon.	As 1 to 1720	38.0	22.0	1490						DAUBENY.
1 Grain to about 10 Quarts.	As 1 to 430	23.5	8.468	35.35			28.619	A trace		THOMSON.
Nearly as strong as No. 1.	———	20.902	12.365	67.78			32.744	0.956		THOMSON.
A trace.	———	19.772	2.121	47.865			40.234	A trace		THOMSON.
A trace.	———	18.777	22.592	26.61			39.457	A trace		THOMSON.

Geological Position.	Locality of the Spring.	Name of the Spring.	Total of its Saline Contents in the Pint.	Iodine. Its proportion to The Water.	Iodine. Its proportion to The Chlorine.
Lias Clay.	Leamington, Warwickshire.	No. 5. Smart's Saline.	92.589	Nearly as the two preceding.	———
	Ditto	No. 6. Lord Aylesford's.	113.995	None detected.	———
	Ditto	No. 7. Reid's Sulphureous.	79.142	None detected.	———
	Ditto*	No. 8. Reid's Saline.	102.597	None detected.	———
	Gloucester.	No. 1. Sulphureous Saline.	84.2	About 1 Grain to 50 Gallons.	As 1 to 12.000
	Ditto	No. 2. Chalybeate Saline.	In all respects agrees with No. 1 in point of com		
	Ditto	No. 3. Strong Saline.	76.5	About 1 Grain to 96 Gallons.	As 1 to 33.000
	Ditto	No. 4. Weak Saline.	75.22	Nearly as No. 3.	———
	Tewksbury.	The Walton Spring.	46.1	About 1 Grain to 36 Gallons.	As 1 to 6690
	Cheltenham.	Pittville, No. 1. "The Pure Saline."	45.8	None detected.	———
	Ditto	Sherborne, No. 4.	84.44	About 1 Grain to 90 Gallons.	As 1 to 33.000
	Ditto	Thomson's, No. 4.	80.13	About 1 Grain to 30 Gallons.	As 1 to 3600
	Ditto	Old Well, No. 1.	81.51	About 1 Grain to 60 Gallons.	As 1 to 19.000
	Ditto*	Thomson's, No. 2.	52.29	None detected.	———
Oolitic Strata.	Melksham, Wiltshire.	The Saline Spring.	107.42	None detected.	———
Chalk Formation.	Epsom, Surry.	The Saline Spring.	33.2	None detected.	———
Tertiary Rocks.	Windsor.	St. Leonard's Hill Spring.		None detected.	———
	Gray's Inn Lane, London.	Chad's Well.		None detected.	———
Present Ocean.	Off Portsmouth.			None detected.	———

* In none of the remaining Cheltenham or Leamington Springs could I satisfy myself of the existence of either iodine or bromine.

Bromine. Its Proportion to The Water.	Bromine. Its Proportion to The Chlorine.	Chlorides of Calcium.	Chlorides of Magnesium.	Chlorides of Sodium.	Sulphates of Lime.	Sulphates of Magnesia.	Sulphates of Soda.	Peroxide of Iron.	Carbonates or other Salts.	Authority on which the Statement of the Saline Ingredients is given.
A trace.	———	17.570	26.05	14.534			34.435	A trace		THOMSON.
A trace.	———	20.561	3.266	40.77			40.398	A trace		THOMSON.
A trace.	———	15.777	9.695	25.60			28.065	A trace		THOMSON.
A trace.	———	17.987	10.813	42.92			30.61	0.265		THOMSON.
About 1 Grain to 10 Quarts.	As 1 to 600			50.41	1.2		10.35		Carbonate of Lime 0.2	DAUBENY.
position, and very nearly in the proportion of its ingredients.										DAUBENY.
Rather less than in No. 1.	As 1 to 860			71.5	2.0		1.6			DAUBENY.
Nearly as No. 3.	———			69.2	2.38		1.15			DAUBENY.
None detected.	———	0.3	1.8	37.5			5.6		Carbonate of Lime 1.0	DAUBENY.
About 1 Grain to 6 Gallons.	As 1 to 768		A trace	27.16			17.55		Carbonate of Lime 0.2	DAUBENY.
None detected.	———	4.29	0.59	72.8			6.76			SCUDAMORE.
None detected.	———	3.07	2.02	46.4			28.64			SCUDAMORE.
None detected.	———	6.21	2.54	58.2			14.56			SCUDAMORE.
A trace, about as much as in the Spring at Pittville.	———	3.31	1.52	25.7			21.76			SCUDAMORE.
A trace.	———	12.0	0.42	90.0						DAUBENY.
A trace.	———			6.0	18.3		3.9		Carbonate of Lime 5.0	DAUBENY.
None detected.	———									
None detected.	———									
1 Grain to 1 Gallon.	As 1 to 840									

XVIII.—*Experiments to determine the difference in the Number of Vibrations made by an Invariable Pendulum in the Royal Observatories of Greenwich and Altona. By Captain* EDWARD SABINE *of the Royal Artillery, Secretary to the Royal Society.*

Read March 25th, 1830.

THE invariable pendulum No. 12, with which these experiments were made, was vibrated in the Royal Observatory at Greenwich in July 1828; in the Royal Observatory at Altona in September and October of the same year; and again at the Royal Observatory at Greenwich in August 1829. The mean of the results obtained at Greenwich, in July 1828 and in August 1829, give the rate of this pendulum at Greenwich, to be compared with its rate obtained at Altona.

The experiments in July 1828 have been already printed in the Phil. Trans. for 1829, Part I. pp. 100-102; the result was as follows:

Therm.	Barom.	Vibrations in Vacuo at 61°.5.
61.50 . . .	29.446 . . .	85970.00

In the experiments in August 1829 (Table A), the same barometer, thermometer, planes, and fixed support were used as in the preceding experiments. The rate of GRAHAM's clock employed in observing coincidences was supplied by Mr. THOMAS GLANVILLE TAYLOR by daily comparison with the Greenwich transit clock. The following is an abstract of the results in Table A.

Greenwich, August 1829; Experiments with Pendulum 12.

	Therm.	Barom.		Vibrations.	Corrected to 63.53.
Aug. 8	63.87	30.102	62.5	85959.22	85959.37
8	65.53	30.072	64.5	85958.41	85959.29
9	62.50	29.988	62.0	85959.76	85959.31
9	64.45	29.951	65.0	85959.24	85959.64
9	64.95	29.908	66.5	85959.04	85959.66
10	62.90	29.806	62.0	85959.63	85959.35
10	63.55	29.829	63.5	85959.29	85959.30
11	61.50	30.020	60.0	85960.27	85959.38
11	62.50	30.047	62.0	85959.79	85959.34
	63.53	29.970	63.0		85959.404

− .090 red^{s}. to 32°. Red^{s}. for 29.9 in. of air at 63°.5 } + 9.926
+ .019 capillary.

29.900 at 32°. Mean 85969.33

The two results then at Greenwich are as follows:

		Vibrations.		Vibrations.
July 1828	61.50	85970.00	Corrected to 62.5 =	85969.56
August 1829	63.53	85969.33	Corrected to 62.5 =	85969.78
	62.5		Mean	85969.67

In Altona the pendulum is vibrated in an apartment on the ground floor of M. SCHUMACHER's house, appropriated to pendulum experiments. The door of this apartment is double, and the windows are double sashed, for the purpose of preserving an uniform temperature: a strong mahogany plank is fixed securely to the wall. The agate planes were screwed on it, and the telescope for observing coincidences was stationed in an aperture made for the purpose in the wall opposite to the pendulum, so that the coincidences were observed without entering the room. The clock for observing coincidences, with which M. SCHUMACHER was so kind as to supply me, was BREGUET, No. 3405, the admirable going of which during five centuries has recently been published. The

rate of this clock during the experiments was furnished by Lieut. Nehus of the Royal Danish Engineers, assistant to M. Schumacher in the Trigonometrical Survey of Denmark; who for that purpose compared it daily with the transit clock of the Altona Observatory at those hours which were not suitable for giving its rate whilst the pendulum was in vibration. The heights of the barometer were furnished by M. Peterson, assistant in the Observatory, from M. Schumacher's standard barometer. The height of the pendulum above the level of the sea in M. Schumacher's house was 15.1 toises.

Having an opportunity on this occasion of comparing the thermometer graduated in 1823 by Mr. Daniell and myself, and employed in the present experiments, as well as in all the former determinations which I have made with invariable pendulums, with a standard thermometer presented by M. Bessel to M. Schumacher, I was anxious to avail myself of it; and M. Schumacher with great kindness undertook himself to make the comparison. The thermometers were compared by immersing the bulbs to the same level in water warmed very gradually in a chronometer stove by means of a small night lamp, and suffered again to cool; the thermometers being compared both in the ascending and descending temperatures. I copy the following particulars of the comparison from M. Schumacher's memorandum.

"B. . . . . are the readings of M. Bessel's thermometer.
E. B. . . are the equations of M. Bessel's thermometer.
R. T. . . are the real temperatures.
S. . . . . are the readings of M. Sabine's thermometer.
E. S. . . are the equations of M. Sabine's thermometer.

1828. Oct. 20.

h	m		S.	B.	E. B.	R. T.	E. S.
1	38	P.M.	59.6	60.0	−0.20	59.80	+0.20
1	43		60.0	60.5	−0.20	60.30	+0.30
2	33		65.0	65.3	−0.23	65.07	+0.07
3	24		70.2	70.6	−0.26	70.34	+0.14
5	0		78.7	81.1	−0.32	80.78	
7	0		86.35	86.7	−0.38	86.32	−0.02
9	25		91.75	92.1	−0.44	91.66	−0.09
11	47		94.60	95.3	−0.47	94.83	+0.23

Oct. 21.

h m	S.	B.	E. B.	R. T.	E. S.
7 18 A.M.	71.1	71.4	−0.27	71.13	+0.03
8 35	68.9	69.1	−0.25	68.85	−0.05
10 23	66.0	66.3	−0.24	66.06	+0.06
11 5	65.25	65.5	−0.23	65.27	+0.02
1 10 P.M.	63.1	63.4	−0.22	63.18	+0.08
3 47	61.1	61.3	−0.21	61.09	−0.01
4 03	61.0	61.2	−0.21	60.99	−0.01
4 22	60.75	61.0	−0.21	60.79	+0.04

Oct. 22.

6 38 A.M.	56.15	56.3	−0.18	56.12	−0.03
7 53	55.8	56.1	−0.18	55.92	+0.12
8 20	55.7	56.0	−0.18	55.82	+0.12
10 40	55.5	55.7	−0.18	55.52	+0.02
11 13	55.4	55.6	−0.18	55.42	+0.02
3 55 P.M.	56.25	56.55	−0.18	55.37	+0.12

The freezing point was found quite correct in pounded melting ice."

On examining the particulars of this comparison, it may be fairly inferred that between the limits compared, that is, between the temperatures of 55° and 90°, the indications of the two thermometers agree every where to less than a tenth of a degree; this agreement is the more satisfactory as it includes a part of the scale of my thermometer, in which the graduation by Mr. DANIELL and myself differed a whole degree from the original graduation of the maker. It is probable that M. BESSEL's thermometer indicates temperatures more often higher than mine, by a few hundredths of a degree, than lower by the same small quantities; hundredths of a degree are, however, quantities to be spoken of with much confidence, and may safely be neglected on the present occasion.

The experiments with the pendulum at Altona are given in Table B; and the following is an abstract of the results.

Altona; September and October 1828. Experiments with Pendulum 12.

	Therm.	Barom.	Vibrations.	Corrected to 58.32.
Sept. 20	60.93	30.267	85968.46	85969.61
20	60.95	30.267	85968.47	85969.63
20	60.52	30.243	85968.54	85969.51
20	60.42	30.219	85968.54	85969.46
21	59.63	30.139	85968.91	85969.49
21	60.10	30.135	85968.74	85969.52
21	60.33	30.115	85968.81	85969.69
22	60.55	29.855	85968.64	85969.62
22	60.60	29.873	85968.67	85969.67
23	60.71	29.994	85968.73	85969.78
Oct. 1	60.17	29.566	85969.16	85969.97
1	60.30	29.578	85969.17	85970.04
2	60.35	29.688	85969.18	85970.07
2	60.02	29.688	85969.29	85970.04
3	59.40	29.976	85969.45	85969.93
4	58.55	29.913	85969.62	85969.72
8	57.50	29.381	85970.19	85969.83
8	56.70	29.433	85970.49	85969.78
15	55.05	30.030	85971.25	85969.81
15	54.60	30.030	85971.35	85969.71
21	53.70	30.195	85971.89	85969.86
22	54.03	30.041	85971.74	85969.86
23	55.02	30.041	85971.29	85969.84
23	55.40	29.835	85971.10	85969.82
24	55.40	29.982	85971.20	85969.92
24	55.43	29.982	85971.20	85969.93
	58.32	29.941	85969.77	85969.77

Red^{n}. for 29.94 inches of air at 58.32 + 10.07

85979.84

The rate at Greenwich, with which this rate at Altona is to be compared, is 85969.67 vibrations at 62°.5. Reducing both these rates to a mean temperature of 60°, in the proportion of 0.44 vibration to one degree of FAHRENHEIT, they become respectively 85979.77 at Greenwich, and 85979.10 at Altona: whence we have an acceleration at Altona of 8.33 vibrations per diem.

TABLE A.—Vibrations of Pendulum 12 at Greenwich, August 1829; on the fixed iron support in the Pendulum room. The Barometer employed was the standard barometer of the Royal Observatory; the Thermometer was Captain SABINE's standard thermometer; the Arc was the same that had been used at Greenwich in July 1828; and at Altona in September and October 1828.

EXP. 1. August 8th. Clock making 86309.45 Vibrations. Barom. {30.119 / 30.085} 30.102 in. 62°.5.

No. of Coincid.	Therm.	Times of Disapp.	Times of Reapp.	Times of Coincidence.	Arc.	Mean Therm.	Mean Interval.	Correction for Arc.	Vibrations in 24 hours.
	°	m s	m s	h m s	°	°	s	s	
1	61.3	27 19	27 23	7 27 21	0.85	63.87	492.46	+0.30	85959.22
33	66.0	.....	.....	12 0 0					
45	64.3	28 22	28 36	1 28 29	0.12				

EXP. 2. August 8th. Clock making 86309.73 Vibrations. Barom. {30.085 / 30.060} 30.072 in. 64°.5.

	°	m s	m s	h m s	°	°	s	s	
1	64.4	37 11	37 17	1 37 14	0.86	65.53	490.89	+0.31	85958.41
	67.1	.....	.....	3 0 0					
41	65.1	4 14	4 45	7 4 29.5	0.14				

EXP. 3. August 9th. Clock making 86309.73 Vibrations. Barom. {30.004 / 29.973} 29.988 in. 62°.

	°	m s	m s	h m s	°	°	s	s	
1	61.7	4 51	4 57	8 4 54	0.72	62.5	492.84	+0.30	85959.76
29	63.3	54 47	55 00	11 54 53.5	0.20				

EXP. 4. August 9th. Clock making 86309.70 Vibrations. Barom. {29.973 / 29.930} 29.951 in. 65°.

	°	m s	m s	h m s	°	°	s	s	
1	63.4	3 25	3 30	12 3 27.5	0.80	64.45	490.0	+0.39	85959.24
27	65.5	36 34	36 45	3 36 39.5	0.23				

TABLE A. (Continued.)

EXP. 5. August 9th. Clock making 86309.70 Vibrations. Barom. {29.930 / 29.886} 29.908 in. 66°.5.

No. of Coincid.	Therm.	Times of Disapp.	Times of Reapp.	Times of Coincidence.	Arc.	Mean Therm.	Mean Interval.	Correction for Arc.	Vibrations in 24 hours.
		m s	m s	h m s	°		s	s	
1	65.4	44 57	45 2	3 44 59.5	0.82	64.95	491.67	+0.43	85959.04
31	64.5	50 41	50 58	7 50 49.5	0.22				

EXP. 6. August 10th. Clock making 86309.70 Vibrations. Barom. {29.804 / 29.808} 29.806 in. 62°.

		m s	m s	h m s	°		s	s	
1	62.8	37 17	37 24	7 37 20.5	0.91	62.9	492.59	+0.38	85959.63
39	63.0	49 12	49 26	12 49 19	0.18				

EXP. 7. August 10th. Clock making 86309.75 Vibrations. Barom. {29.808 / 29.850} 29.829 in. 63°.5.

		m s	m s	h m s	°		s	s	
1	63.2	57 49	57 53	12 57 51	0.84	63.55	492.15	+0.39	85959.29
44	63.9	50 19	50 48	6 50 33.5	0.12				

EXP. 8. August 11th. Clock making 86309.75 Vibrations. Barom. {30.000 / 30.040} 30.020 in. 60°.

		m s	m s	h m s	°		s	s	
1	61.0	21 51	21 55	8 21 53	0.97	61.5	493.29	+0.47	85960.27
36	62.0	9 31	9 45	1 9 38	0.19				

EXP. 9. August 11th. Clock making 86309.75 Vibrations. Barom. {30.040 / 30.053} 30.047 in. 62°.

		m s	m s	h m s	°		s	s	
1	62.4	18 19	18 25	1 18 22	0.84	62.5	492.88	+0.39	85959.79
42	62.6	54 52	55 28	6 55 10	0.12				

TABLE B.—Vibrations of Pendulum 12 at Altona; September 1828.

EXP. 1, 2, 3, 4. September 20th. Clock making 86436.48 Vibrations.

No. of Coincid.	Therm.	Times of Disapp.	Times of Reapp.	Times of Coincidence.	Arc.	Mean Therm.	Mean Interval.	Correction for Arc.	Vibrations in 24 Hours.	Barom.
	°	m s	m s	h m s	°	°	s	s		°
1	60.5	43 43	43 49							
2	.....	49 52	49 58	9 49 55.33	0.515					
3	60.9	56 2	56 5			60.93	369.17	+0.24	85968.46	30.267 61
	61.0	.....	.....	11						
24	61.1	5 13	5 21	12 5 17	0.250					
	61.1	.....	.....	1		60.95	369.32	+0.05	85968.47	30.267 61
50	60.8	45 4	45 35							
51	.....	51 13	51 45	2 51 28.67	0.100					
52	60.8	57 21	57 54							
	Fresh impulse.									
1	60.7	10 25	10 29							
2	.....	16 33	16 39	3 16 37.17	0.560					
3	60.8	22 42	22 49			60.52	369.25	+0.22	85968.54	30.243 61
27	60.3	50 12	50 43	5 50 27.5	0.210					
	Fresh impulse.									
1	60.8	46 55	47 5							
2	.....	53 5	53 14	7 53 9.17	0.575					
3	60.8	59 14	59 22							
38	60.0	34 32	34 52			60.42	369.25	+0.20	85968.54	30.219 60
39	.....	40 40	41 1	11 40 50.83	0.165					
40	60.1	46 49	47 11							

EXP. 5, 6, 7. September 21st. Clock making 86436.75 Vibrations.

	°	m s	m s	h m s	°	°	s	s		°
1	59.3	15 21	15 28	7 15 24.5	0.570	59.63	369.275	+0.29	85968.91	30.139 61
26	59.8	18 22	18 38	9 18 30.5	0.290					
39	60.4	9 11	9 24	11 9 17.5	0.160	60.10	369.306	+0.08	85968.74	30.135 61
	Fresh impulse.									
1	60.4	33 38	33 42	11 33 40	0.510					
	60.4	.....	.....	1 0 0		60.33	369.292	+0.17	85968.81	30.115 61
37	60.2	15 4	15 25	3 15 14.5	0.160					

EXP. 8 and 9. September 22nd. Clock making 86436.66 Vibrations.

	°	m s	m s	h m s	°	°	s	s		°
1	59.5	40 41	40 47							
2	.....	46 50	46 56	9 46 53	0.535					
3	60.0	52 59	53 5							
	60.9	.....	.....	11 14		60.55	369.216	+0.18	85968.64	29.855 60
38	61.0	28 18	28 31							
39	.....	34 26	34 43	1 34 34	0.160					
40	61.0	40 35	40 51							
	Fresh impulse.									
1	61.0	46 36	46 40							
2	.....	52 45	52 50	1 52 47.33	0.595					
3	.....	58 54	58 59							
	60.6	.....	.....	3 50		60.60	369.246	+0.17	85968.67	29.873 60
45	.....	17 13	17 35							
46	.....	23 27	23 41	6 23 34.17	0.120					
47	60.2	29 30	29 59							

TABLE B. (Continued.)

EXP. 10. September 23rd. Clock making 86436.85 Vibrations.

No. of Coincid.	Therm.	Times of Disapp.	Times of Reapp.	Times of Coincidence.	Arc.	Mean Therm.	Mean Interval.	Correction for Arc.	Vibrations in 24 Hours.	Barom.
	°	m s	m s	h m s	°					
1	.....	21 46	21 48							
2	60.25	27 54	27 57	9 27 55.83	0.645					
3	.....	34 3	34 7							
	60.9	.....	.....	11						
	60.9	.....	.....	1		°60.71	s 369.142	s +0.19	85968.73	29.994 ° 60
48	.....	10 44	11 9							
49	60.8	16 54	17 16	2 17 5.5	0.115					
50	.....	23 5	23 25							

EXP. 11 and 12. October 1st. Clock making 86437.45 Vibrations.

	°	m s	m s	h m s	°					
1	59.95	51 7	51 11	9 51 9	0.60					
31	.....	55 28	55 49							
32	60.4	1 36	1 57	1 1 47	0.20	°60.17	s 368.97	s +0.24	85969.16	29.566 ° 61
33	.....	7 46	8 6							
	Fresh impulse.									
1	60.4	14 25	14 31	1 14 28	0.590					
32	.....	24 57	25 16			°60.3	368.98	+0.23	85969.17	29.578 61
33	60.2	31 3	31 27	4 31 15.33	0.200					
34	.....	37 14	37 35							

EXP. 13 and 14. October 2nd. Clock making 86437.70 Vibrations.

		m s	m s	h m s	°					
1	.....	40 52	40 57							
2	60.05	47 0	47 5	9 47 3	0.510					
3	.....	53 9	53 15							
	60.70	.....	.....	11		°60.35	s 368.83	s +0.19	85969.18	29.688
31	.....	45 7	45 31							
32	60.30	51 19	51 37	12 51 28	0.190					
33	.....	57 30	57 44							
	Fresh impulse.									
1	60.20	4 38	4 44	1 4 41	0.550					
33	.....	21 15	21 36			°60.025	368.91	+0.21	85969.29	29.688
34	59.85	27 23	27 47	4 27 35	0.190					
35	.....	33 32	33 57							

EXP. 15. October 3rd. Clock making 86437.58 Vibrations.

	°	m s	m s	h m s	°					
1	59.25	58 20	58 23							
2	.....	4 29	4 32	10 4 30.33	0.610					
3	59.45	10 37	10 41							
	59.70	.....	.....	11 0		°59.40	s 369.089	s +0.25	85969.45	29.976
31	59.30	2 46	3 2							
32	.....	8 55	9 11	1 9 3	0.210					
33	59.30	15 4	15 20							

TABLE B. (Continued.)

EXP. 16. October 4th. Clock making 86437.43 Vibrations.

No. of Coincd.	Therm.	Times of Disapp.	Times of Reapp.	Times of Coincidence.	Arc.	Mean Therm.	Mean Interval.	Correction for Arc.	Vibrations in 24 Hours.	Barom.
	°	m s	m s	h m s	°	°	s	s		
1	58.55	38 55	38 59	10 38 57	0.670	58.55	369.32	+0.28	85969.62	29.913
36	.....	14 12	14 52							
37	58.55	20 22	20 43	2 20 32.5	0.200					
38	.....	26 32	26 54							

EXP. 17. October 8th. Clock making 86437.51 Vibrations.

	°	m s	m s	h m s	°	°	s	s		
1	57.6	44 44	44 50	2 44 47	0.580	57.5	369.75	+0.20	85970.19	29.381
39	.....	38 48	39 7							
40	57.4	44 57	45 17	6 45 7.17	0.160					
41	.....	51 7	51 27							

EXP. 18. October 9th. Clock making 86437.51 Vibrations.

	°	m s	m s	h m s	°	°	s	s		
1	56.8	54 14	54 19	9 54 16.5	0.535	56.7	370.014	+0.19	85970.49	29.433
34	.....	17 40	17 54							
35	56.6	23 50	24 5	1 23 57	0.180					
36	.....	30 1	30 12							

EXP. 19 and 20. October 15th. Clock making 86437.67 Vibrations.

	°	m s	m s	h m s	°	°	s	s		
1	.....	12 13	12 16							
2	55.2	18 23	18 27	1 18 24.67	0.60	55.05	370.44	+0.26	85971.25	30.030
3	.....	24 33	24 36							
28	54.9	58 49	59 3	3 58 56	0.23					
	Fresh impulse.									
1	54.9	23 45	23 49	4 23 47	0.58	54.6	370.57	+0.21	85971.35	30.030
35	.....	53 38	53 55							
36	54.3	59 47	0 7	7 59 56.83	0.19					
37	.....	5 56	6 18							

EXP. 21. October 21st. Clock making 86437.62 Vibrations.

	°	m s	m s	h m s	°	°	s	s		
1	54.0	31 3	31 8	12 31 5.5	0.59	53.7	371.04	+0.21	85971.89	30.195
36	53.4	7 20	7 44	4 7 32	0.18					

EXP. 22. October 22nd. Clock making 86437.56 Vibrations.

	°	m s	m s	h m s	°	°	s	s		
1	53.9	29 15	29 17	11 29 16	0.63	54.03	370.94	+0.24	85971.74	30.041
	54.0	.....	.....	12 30						
39	.....	24 0	24 22							
40	54.2	30 13	30 34	3 30 22.67	0.18					
41	.....	36 22	36 45							

TABLE B. (Continued.)

EXP. 23 and 24. October 23rd. Clock making 86437.34 Vibrations.

No. of Coincid.	Therm.	Times of Disapp.	Times of Reapp.	Times of Coincidence.	Arc.	Mean Therm.	Mean Interval.	Correction for Arc.	Vibrations in 24 Hours	Barom.
	°	m s	m s	h m s	°	°	s	s		
1	54.4	39 51	39 56	9 39 53.5	0.60					
	55.1	.....	.....	10 40		55.025	370.76	+0.23	85971.29	30.041
	55.3	.....	.....	11 50						
35	55.3	9 50	10 9	1 9 59.5	0.19					
		Fresh impulse.								
1	55.3	16 28	16 32	1 16 30	0.61	55.4	370.50	+0.34	85971.10	29.835
19	55.5	7 34	7 44	3 7 39	0.32					

EXP. 25 and 26. October 24th. Clock making 86437.45 Vibrations.

No. of Coincid.	Therm.	Times of Disapp.	Times of Reapp.	Times of Coincidence.	Arc.	Mean Therm.	Mean Interval.	Correction for Arc.	Vibrations in 24 Hours	Barom.
	°	m s	m s	h m s	°	°	s	s		
1	55.4	25 12	25 16	10 25 14	0.60	55.4	370.59	+0.24	85971.20	29.982
33	55.4	42 45	43 1	1 42 53	0.20					
		Fresh impulse.								
1	55.4	49 50	49 52	1 49 51	0.61					
	55.5	.....	.....	4		55.43	370.57	+0.26	85971.20	29.982
28	55.4	36 25	36 48	4 36 36.5	0.22					

XIX. *Experiments to ascertain the Correction for Variations of Temperature, within the limits of the natural temperature of the Climate of the South of England, of the Invariable Pendulum recently employed by British observers.* By *Captain* Edward Sabine *of the Royal Artillery, Secretary to the Royal Society.*

Read March 25th, 1830.

HAVING obtained the rate of the invariable pendulum, No. 12, in the Royal Observatory at Greenwich at the temperature of 63°.53 in the month of August 1829, as given in the preceding paper, the pendulum was laid aside until the cold weather of the winter season had set in, when the observations detailed in Table C were made with it, at the mean temperature of 31°.15; the results of which are exhibited in the following abstract:

Greenwich, December 1829, and January 1830: Experiments with Pendulum 12.

	Therm.	Barom.		Vibrations.	Corrected to 31.15.
Dec. 29	28.50	30.212	27.0	85973.99	85972.82
29	28.55	30.227	27.0	85973.84	85972.70
30	29.30	30.307	28.0	85973.66	85972.85
30	29.65	30.311	28.5	85973.52	85972.86
Jan. 1	31.35	30.497	30.5	85972.82	85972.91
1	32.00	30.472	31.0	85972.33	85972.70
2	31.80	30.371	31.0	85972.46	85972.75
3	33.85	30.285	33.0	85971.78	85972.97
5	35.40	30.149	34.5	85970.94	85972.81
		30.315	30		85972.81
		−.002 Red^{n}. to 32°		Red^{n}. for 30.33 in. of air at 31.15	+10.77
	31.15	+.019 Capillary			
		30.332 at 32°			85983.58 at 31.15

Comparing this result with 85969.33 vibrations at the temperature of 63°.53 obtained in August 1829, we have 14.25 vibrations per diem corresponding to 32°.38 of FAHRENHEIT; which gives a correction of 0.44 of a vibration per diem for each degree of FAHRENHEIT between 30° and 60°.

The experiments which I formerly made with two pendulums similar to the present, in a chamber artificially heated to between 80° and 90°, gave for the correction for each degree of FAHRENHEIT, respectively for the two pendulums, 0.432 and 0.430, corresponding to that part of the thermometer scale which is included between 45° and 85°. Those results accord well with each other, and are somewhat different from that which is now obtained, and which correspond to the part of the scale comprised between 30° and 60°. But in the experiments in the chamber artificially heated, the fluctuations of temperature, in spite of every precaution, were considerable, and rendered the determination of the mean temperature more difficult, and probably less exact than in the natural temperatures: hence it would be unsafe to conclude in favour of the inference to which these facts would otherwise lead, that the correction at high temperatures is less than at low temperatures, or that the metal expands a smaller proportion of its length for one degree between 85° and 45°, than for one degree between 60° and 30°.

The experiments at Greenwich were made in those extremes of natural temperature afforded by the climate, in which a tolerably uniform temperature is maintained for several days; a condition requisite in such delicate determinations. The clock by GRAHAM was going well on both occasions, and its rate was assigned by Mr. TAYLOR from comparisons with the transit clock of the Observatory, with probably as much accuracy as the rates of clocks are ever obtained. The thermometer is entitled, by its comparison with those of Mr. DANIELL and of M. BESSEL, to be regarded as one of good authority, having, in the absence of a standard thermometer in this country, received the sanction of what must be considered the best existing authorities. Every precaution was adopted, which the experience I have had in obtaining the rates of pendulums has suggested, for maintaining a uniform temperature in the apartment. The examination of the partial observations will best show the success of these precautions. Viewing all these particulars, I regard 0.44 of a vibration per diem for each degree of FAHRENHEIT as a result obtained under circumstances

of a very favourable nature, and as not likely to be surpassed in the confidence which may be due to it, until by a better command of artificial temperatures the experiments can be made to include, with tolerable certainty of determination, a greater difference of temperature than is afforded by the natural climate of this country.

As the many invariable pendulums which have been employed of late years by British observers have all been made of the same kind of brass, it is probable that the same correction for temperature will apply equally to all.

When, as in ordinary cases, the differences of temperature between observations designed to be comparative amount only to a few degrees, the probable error which may be incurred by employing the correction 0.44 for each degree as now determined, can only be very inconsiderable: but when the differences of temperature amount to 50°, which is a case of actual experience in pendulum observations, the question of whether 0.43 or 0.44, for example, be the more correct value, involves an uncertainty in the ultimate result of no less than half a vibration a day. It seems therefore desirable, for the sake of experiments, which are becoming greatly multiplied, and which are daily increasing in accuracy, that means should be devised of obtaining the rates of pendulums in artificial temperatures, embracing a wider range than the natural temperatures, but capable of being determined with equal accuracy.

TABLE C.—Vibrations of Pendulum 12 at Greenwich, December 1829, and January 1830, on the fixed iron support in the Pendulum room. The Barometer employed was the standard barometer of the Royal Observatory; the Thermometer was Captain SABINE's standard thermometer; the Arc was distant 60 inches from the point of suspension, and was divided into degrees, each of 0.8 of an inch in length; the registered arc therefore multiplied by 0.764, gives degrees of the true arc of vibration.

EXP. 1. Dec. 29th. Clock making 86552.75 Vibrations. Barom. $\left\{\begin{matrix}30.205\\30.220\end{matrix}\right\}$ 30.212 27°.

No. of Coincid.	Therm.	Times of Reapp.	Times of Disapp.	Times of Coincidence.	Registered Arc.	True Arc.	Mean Therm.	Mean Interval.	Correction for Arc.	Vibrations in 24 hours Mean Solar Time.
	°	m s	m s	h m s	°	°	°	s	s	
1	28.6	15 37	15 42	1 15 39.5	0.84	0.64	28.5	298.97	+0.24	85973.99
53	28.4	34 38	34 54	5 34 46	0.24	0.18				

TABLE C. (Continued.)

EXP. 2. Dec. 29th. Clock making 86552.75 Vibrations. Barom. {30.220 / 30.234} 30.227 27°.

No. of Coincid.	Therm.	Times of Reapp.	Times of Disapp.	Times of Coincidence.	Registered Arc.	True Arc.	Mean Therm.	Mean Interval.	Correction for Arc.	Vibrations in 24 hours Mean Solar Time.
	°	m s	m s	h m s	°	°	°	s	s	
1	28.5	40 3	40 9	5 40 6	0.78	0.59	28.55	298.92	+0.20	85973.84
55	28.6	9 1	9 15	10 9 8	0.19	0.15				

EXP. 3. Dec. 30th. Clock making 86552.75 Vibrations. Barom. {30.314 / 30.300} 30.307 28°.

	°	m s	m s	h m s	°	°	°	s	s	
1	29.4	42 27	42 32	10 42 29.5	0.90	0.67	29.30	298.82	+0.22	85973.66
63	29.2	51 9	51 24	3 51 16.5	0.18	0.14				

EXP. 4. Dec. 30th. Clock making 86552.75 Vibrations. Barom. {30.300 / 30.322} 30.311 28°.5.

	°	m s	m s	h m s	°	°	°	s	s	
1	29.5	0 0	0 5	4 0 2.5	0.90	0.67	29.65	298.77	+0.18	85973.52
78	29.8	23 15	23 40	10 23 27.5	0.12	0.09				

EXP. 5. Jan. 1st. Clock making 86552.75 Vibrations. Barom. {30.515 / 30.480} 30.497 30°.5.

	°	m s	m s	h m s	°	°	°	s	s	
1	31.0	17 37	17 40	11 20 7.5	0.98	0.75	31.35	298.36	+0.26	85972.82
2	.....	22 34	22 39							
62	.....	20 52	21 4	4 23 27.25	0.18	0.14				
63	31.7	25 51	26 2							

EXP. 6. Jan. 1st. Clock making 86552.75 Vibrations. Barom. {30.480 / 30.465} 30472 31°.

	°	m s	m s	h m s	°	°	°	s	s	
1	32.0	40 50	40 53	4 40 51.5	0.86	0.64	32.0	298.14	+0.17	85972.33
74	32.0	43 29	43 43	10 43 36	0.13	0.10				

EXP. 7. Jan. 2nd. Clock making 86552.75 Vibrations. Barom. {30.378 / 30.365} 30.371 31°.

	°	m s	m s	h m s	°	°	°	s	s	
1	31.6	37 24	37 27	11 37 25.5	1.01	0.77	31.8	298.17	+0.24	85972.46
76	32.0	50 1	50 15	5 50 8	0.13	0.10				

TABLE C. (Continued.)

EXP. 8. Jan. 3rd. Clock making 86552.75 Vibrations. Barom. {30.300 / 30.270} 30.285 33°

No. of Coincid.	Therm.	Times of Reapp.	Times of Disapp.	Times of Coincidence.	Registered Arc.	True Arc.	Mean Therm.	Mean Interval.	Correction for Arc.	Vibrations in 24 hours Mean Solar Time.
	°	m s	m s	h m s	°	°	°	s	s	
1	33.8	13 27	13 35	2 13 31	0.92	0.68	33.85	297.86	+0.18	85971.78
78	33.9	35 38	35 54	8 35 46	0.11	0.08				

EXP. 9. Jan. 5th. Clock making 86552.75 Vibrations. Barom. {30.110 / 30.188} 30.149 34°.5

	°	m s	m s	h m s	°	°	°	s	s	
1	35.4	16 25	16 31	2 16 28	0.98	0.75	35.4	297.42	+0.20	85970.94
81	35.4	52 52	53 11	8 53 1.5	.12	0.08				

XX. *On a new Register-Pyrometer, for measuring the Expansions of Solids, and determining the higher Degrees of Temperature upon the common thermometric scale.* *By* J. Frederic Daniell, *Esq., F.R.S.*

Read June 17, 1830.

In the year 1821 I published in the Journal of the Royal Institution* an account of a new pyrometer, and the results of some experiments with it, which were the means of correcting the highly erroneous notions which had, up to that time, been generally entertained of the degrees of temperature beyond the boiling point of mercury. The instrument was capable of affording correct determinations, connected in an unexceptionable manner with the scale of the mercurial thermometer; but, although applicable to scientific investigation in careful hands, it could be inserted only into experimental furnaces of a particular construction, which greatly limited its use. The great desideratum still remained of a pyrometer, which might universally be applied to the higher degrees of heat, as the thermometer has long been to the lower; and which, in addition to its use in delicate researches, might effect for the potter, the smelter, the enameler and others, in the routine of their business, what the latter daily performs for the brewer, the distiller, the sugar-refiner, and the chemist.

I shall now have the honour of laying before the Royal Society a description of a contrivance which, I trust, will be found to answer all the desired purposes; and which, while simple enough to be intrusted to the hands of common workmen in every variety of fire-place, I hope to prove, by the results of my experiments, to be sufficiently delicate to extend considerably our knowledge of the expansion of metals, upon which so much labour has been bestowed by some of the first philosophers.

I was not aware, at the time when I wrote the account above referred to, that the subject had been previously investigated by M. Guyton de Morveau, and that he had proposed to apply the expansion of platinum as a measure of high

* Vol. xi. p. 309.

temperature, and more particularly to the purpose of connecting the indications of Wedgwood's pyrometer with the mercurial scale and verifying its regularity. I have since carefully studied his laborious papers in the Annales de Chimie *, and the Mémoires de l'Institut †, which appear to have been but very little known in this country; and previously to entering upon the more particular object of the present paper, I must claim indulgence for a few remarks upon the general state of the inquiry at the time when its pursuit was abandoned by that able philosopher.

M. Guyton's pyrometer consisted of a small bar or plate of platinum 45 millimetres (1.77 inch) long, 5 millimetres (about 0.2 inch) broad, and 2 millimetres (about 0.08 inch) thick, placed in a groove formed in a piece of highly baked porcelain. One extremity of this bar rested upon the solid end, which terminated the groove, and the other pressed upon the short arm of a bent lever, the longer arm of which terminated in a point and moved on a pivot over the graduated arc of a circle; indicating by its motion any lengthening of the bar by increase of temperature. The short arm of the lever was 2.5 millimetres and the long arm 50 millimetres in length, and the latter carried a nonius by which the tenths of a degree might be read off. The whole was constructed of platinum; and a plate of the same metal was made to press, in the manner of a spring, upon the extremity of the index, to prevent any displacement when withdrawing it from the fire. The description of this instrument in the first Essay, published in the year 1803, was not accompanied by any explanatory figure; and the notice in the Annales terminates by announcing that the inventor had at that time only begun "a series of experiments to determine its march, to compare it with the pyrometer pieces of Wedgwood, and to ascertain the degree of confidence which might be placed in the indications of the latter." The second Essay did not appear till the year 1808, and in it M. Guyton observes that "many persons had expressed a wish to be made acquainted with the improvements which he had made in the instrument since its first construction; and that he had determined in consequence to give a fresh description of it accompanied by drawings, which might enable artists who undertook its construction to render it comparable. He, however, thought it right to give a previous account of the labours of others in this branch of science, and

* Tome xlvi. p. 276. † 1808, Second Semestre, tome ix.—1811, ibid. tome xii.

to remove certain errors which had prevailed up to that time concerning the pyrometer then most in use (Wedgwood's), and which might possibly prove most commodious, and consequently most useful, if once the degree of exactitude could be determined of which it was susceptible." The remainder of the paper is taken up with an account of the most accurate experiments upon the expansion of the metals from the time of Newton.

The third and last Essay was delayed till the year 1811; and in it no further description of the platinum pyrometer is to be found; but a laborious comparison,

1st, of the indications of the platinum pyrometer with those of the mercurial thermometer;

2nd, of the same pyrometer with that of Wedgwood; and

3rd, of the degrees determined by these instruments with those previously known of the expansion, ebullition and fusion of various substances; in a range of temperature comprising the highest degrees of the thermometric scale and the lowest of Wedgwood's.

Now it is very remarkable that all M. Guyton's efforts in this paper are directed to the valuation of the degrees determined by Mr. Wedgwood's clay pieces; but that he carries the comparison of the platinum pyrometer by actual experiment no higher than the melting point of antimony. He clearly establishes a great error in Mr. Wedgwood's original estimation of his degrees to that point; and, by calculation upon this basis, continues the correction to the melting point of iron, "en admettant toujours une progression uniforme jusque dans les plus hautes températures." The experimental comparison was obviously stopped by some practical difficulty at higher temperatures; and it is easy to perceive in what this must have consisted. Platinum at a red heat becomes very soft and ductile; and the lever against which the pyrometric bar pressed, being of such very slender dimensions, would obviously be liable to bend and thus frustrate the experiment: in addition to which, I can speak from my own experience, the platinum spring plate and the centre pin would be liable to a change of texture which would impede the motion of the lever and it would finally become welded to the index; for a very moderate pressure at a high temperature would produce this effect.

The conclusion, indeed, of these Essays seems to admit that the author did

not expect that the platinum pyrometer could ever come into general use: "enfin, ces corrections ne peuvent manquer d'ajouter à l'utilité du pyromètre d'argile, soit dans les travaux chimiques, soit dans les arts; quand même le pyromètre de platine, plus exact mais moins usuel, serait réservé pour en assurer la marche, et pour servir à des recherches plus importantes."

M. Guyton, however, although he abundantly proves the incorrectness of Mr. Wedgwood's estimate of the higher degrees of temperature, is very far indeed from establishing the point at which he so earnestly laboured, namely, the regularity of the contraction of the clay pieces; or from substituting a more correct value of the degrees throughout the whole range of the gauge than the one which he so completely overturned. His comparative experiments with the platinum pyrometer, at the boiling points of mercury and linseed oil, and the melting point of antimony, led him to reduce the equivalent of each degree from 130° Fahr. to 62°.5. The zero point of the clay pyrometer was thus carried back to 517° instead of 1077°; but it seems to have escaped his notice that this zero point was declared to be a red heat visible in the day-light,—a description which cannot be mistaken, and which clearly could not be below the temperature of boiling oil, melting lead, or boiling mercury; all of which are, however, placed above it in M. Guyton's table. M. Guyton also places the melting point of silver at the 22nd degree of Mr. Wedgwood's scale instead of the 28th, which was, according to his own determination, a correction first suggested by Sir James Hall in the 9th volume of Nicholson's Journal. Taking the value of each degree at 62°.5 Fahr., it fixes this point at 1892° Fahr., which agrees very nearly with my own experiment in the paper before alluded to; but continuing the calculation up to the melting point of iron, upon the supposition of an uniform progression, the 130th degree corresponds with 8696° Fahr., which, although only about half the amount 17977° assigned by Mr. Wedgwood, is very far removed from the result of my calculation 3479°.

Nevertheless, it is a curious fact, that M. Guyton's Essay contains proof that his determination is erroneous, and that mine is a near approximation to the truth. As a collateral means of verifying the indications of instruments intended to measure high degrees of temperature, he refers to the calorimeter as capable of affording the necessary data by a calculation from the amount of heat communicated to known quantities of fire or water by bodies in a state of

incandescence; and he quotes the very exact experiments of MM. CLEMENT and DESORMES, who had in this manner determined the following points:

	By the Liquefaction of Ice. Fahr.	By the Heat communicated to Water. Fahr.
Temperature of soft iron melted	3988°	3902°
Cast iron just on the point of melting	3164°	——
Red hot iron	2732°	——
White hot ditto	3282°	——
Iron just ceasing to be luminous in day-light	——	1272°
Melted copper	——	2294°

My own determinations of the melting point of cast iron, 3479°, of that of copper, 2548°, and of a red heat, about 1000°, agree, very closely and satisfactorily with these results, with which I was unacquainted at the time of my experiments. M. GUYTON's remark upon the latter is: "Il suffit de jeter un coup-d'œil sur les résultats, pour recueillir de nouvelles preuves univoques de la nécessité de réduire les valeurs données par WEDGWOOD aux degrés de son pyromètre. Mais je ne crains pas de dire que ces réductions sont ici portées trop loin, ainsi qu'on peut en juger en les rapprochant de celles auxquelles j'ai été conduit par l'ensemble des expériences rapportées dans cet essai. Ce n'est pas que je veuille répandre des doutes sur l'exactitude des observations dont je dois la communication aux deux habiles chimistes ci-dessus cités; mais il est aisé de faire voir que la différence des résultats est due, pour la plus grande partie, à la différence des procédés; de sorte que les évaluations qu'ils ont données aux degrés de l'échelle de WEDGWOOD, peuvent, en dernière analyse, et en prenant les termes moyens dans la latitude que comportent des opérations aussi délicates, servir plutôt à confirmer qu'à détruire le système de correction que j'ai établi."

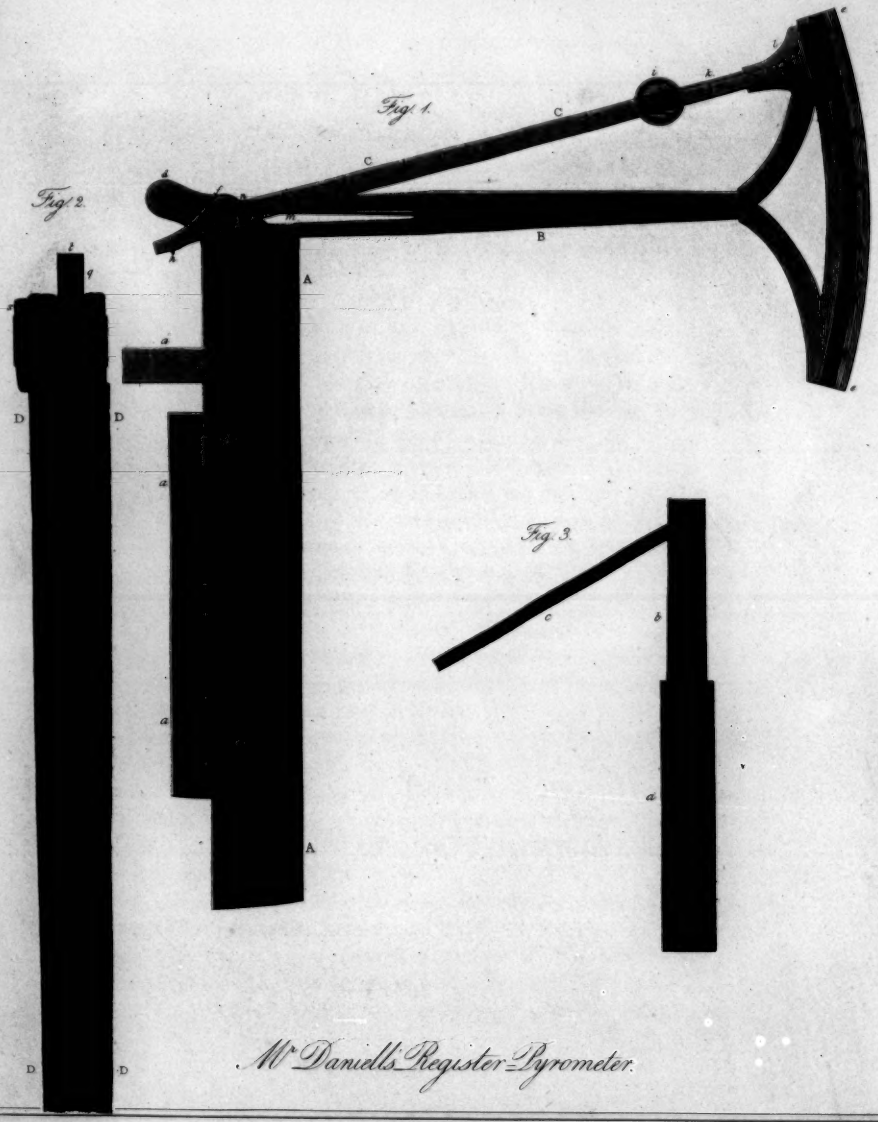
It is worthy of observation, that had the degrees of WEDGWOOD's pyrometer been valued from this determination of the fusing point of iron, the result would have better corresponded with the whole series of phenomena. Instead of 130° FAHR. as fixed by the inventor, or 62°.5 as corrected by M. GUYTON, they would have been estimated at about 20° FAHR.; and taking Mr. WEDGWOOD's original determination of the fusing point of silver at 28° of his scale and the zero point at 1077°, the former would come out about 1650°. By raising the zero point a little, (and it is much more probable that the temperature of a red heat fully

visible in the day-light is above 1077° than below it,) we arrive at something like an approximation to the truth. These wide discrepancies, and the practical disuse of both Mr. Wedgwood's and M. Guyton's pyrometers for a long time past, prove the expediency of further investigating a subject of so much interest and importance.

The pyrometer, which I shall now proceed to submit to the judgement of the Society, consists of two distinct parts, which I shall designate as the Register and the Scale.

The first is a solid bar of black-lead earthenware, eight inches long, seven-tenths of an inch wide, and of the same thickness, cut out of a common black-lead crucible. In this a hole is drilled three-tenths of an inch in diameter, and $7\frac{1}{2}$ inches deep. At the upper end of this bar and on one of its sides about six-tenths of an inch in length of its substance is cut away to the depth of half the diameter of the bore. When a bar of any metal $6\frac{1}{2}$ inches long is dropped into this cavity, it rests against its solid end; and a cylindrical piece of porcelain about $1\frac{1}{2}$ inch long, which I shall call the index, is placed upon the top of it, which projecting into and beyond the open part, is firmly confined to its place by a ring, or strap of platinum; which passing round the black-lead bar and over the piece of porcelain, is made to press upon the latter with any required degree of tension by means of a small wedge of porcelain inserted between the bar and the strap on the side of the former. It is obvious that when such an arrangement is exposed to a high temperature, the metallic bar will force the index forward to the amount of the excess of its expansion over that of the black-lead, and that when again cooled, it will be left at the point of greatest elongation. It may also be observed, that the exact indication of this amount is not in the slightest degree interfered with by any permanent contraction which the black-lead may undergo at high degrees of heat; as any such contraction will take place at the moment of the greatest expansion of the metal, and the index will still mark its point of furthest extension upon this contracted basis.

The problem now consists in the accurate measurement of the distance which the index has been thrust forward from its original position; and although the amount can in any case be but small, there is no reason why it may not be determined with the same precision as is now commonly attained in similar quantities in astronomical and geodetical operations. For this purpose the



Mr Daniell's Register-Pyrometer.

scale is constructed of two rules of brass, accurately joined together at a right angle by their edges, and fitting square upon two sides of the black-lead bar, and of about half its length. At one end of this double rule a small plate of brass projects at a right angle, which plate, when the two sides of the former are applied to the two sides of the register, is brought down upon the shoulder formed by the notch cut away at its upper end, and the whole may be thus firmly adjusted to the black-lead bar by three planes of contact.

On the outside of this frame another brass rule is firmly screwed down, which projecting beyond it, and bending a little so as to bring its end opposite to the cavity in the black-lead bar when applied to it, supports a moveable arm exactly $5\frac{1}{2}$ inches long, turning at its fixed extremity upon a centre, and at its other carrying an arc of a circle, accurately divided into degrees and thirds of a degree, whose radius is exactly 5 inches. At the centre of this circle upon the arm, and of course at the distance of half an inch from the centre of motion, another lighter arm is made to turn, one end of which, being the exact radius of the circular arc, carries a nonius with it, which moves upon the face of the arc and subdivides the former graduation into minutes. The other end crosses the centre; and at the exact distance of one-tenth of the radius, or the distance between the two centres of motion, terminates in an obtuse steel point turned inwards at a right angle. These graduations and distances are laid down with the greatest precision by Mr. Troughton's dividing engine. This part of the apparatus may be regarded as a pair of proportional compasses attached to the end of the brass rule and frame, whose longer legs carrying the arc and nonius are to its shorter as ten to one; and the opening of the latter being regarded as a chord of a small circle, is magnified in the same proportion by the former, and measured upon the scale. A small steel spring let into the larger arm is made to press upon the smaller, so as to adjust the nonius to the commencement of the graduation; and when forced back it tends to restore it to its original position.

The annexed figures, in which all the parts are drawn of their real dimensions, will assist the comprehension of the preceding description. Plate X. fig. 1. represents the scale. A A is the principal brass rule, upon the under side of which the frame *a a a a a a'* is adjusted by the screws *b b*, and which supports upon its bent extremity *c*, the arm B moving upon the centre *d*, and terminating in the arc of the circle *e e*.

C C is the lighter arm moving upon the centre *f* upon the arm B, and carrying at one end the nonius *g*, and at the other the steel point *h*, the distance of which from the centre *f* is exactly half an inch or one-tenth of the radius *f g*, and equal to the distance of the two centres *f d*. *i* is a small lens represented as lying down, but which may be raised by the centres *k* and *l* perpendicularly above the nonius to facilitate the reading. *m m* is the steel spring, which being fixed in a cavity cut out of the arm B, presses upon a small pin *n* on the arm C, and throws the radius back to the commencement of the arc.

Fig. 2. represents the register. D D D D is the black-lead bar, with its cavity *o o*. At *p p p p* it is cut away to the depth of half the bore. *q q* is the porcelain index, which is placed upon the top of the metallic bar, and confined to its place by the pressure of the platinum strap *r* acting by the force of the small porcelain wedge *s*.

When an observation is to be made, the metallic bar is placed in the cavity of the register, the index is to be pressed down upon it and firmly fixed in its place by the platinum strap and porcelain wedge. The scale is then to be applied by carefully adjusting the brass rules to the sides of the black-lead bar, and fixing it by pressing the cross piece (*a'*) upon the shoulder: holding the whole together steadily in the left hand, the moveable arm should be so placed that the steel point (*h*) of the other leg of the compasses may rest upon the edge of the porcelain index, against which it will be pressed with some force by the spring: then moving the arm gently forward with the right hand, the point will slide along the end of the index till it drops into a small cavity (*t*) formed for its reception, and which exactly coincides with the axis of the metallic bar in the register, and the centre of motion of the compasses on the brass rule. The minute of the degree must then be noted, which the nonius indicates upon the arc. A similar observation must be made after the register has been exposed to an increased temperature and again cooled; and the number of degrees or minutes which the nonius will then mark will, by a simple calculation from the known length of the radii and angle, give the length of the chord comprised between the original position of the compasses and the point to which they have moved, or the distance which the index has been forced forward. Such an operation appears complex in the description, but is in fact extremely simple after a little practice, and does not require more than a few seconds for its performance. The scale of this pyrometer being completely

detached from the part which is exposed to the fire, obviates one important objection which has always been made to other contrivances of the same nature, from the uncertain degree of heat and expansion to which they are liable; while the simplicity of that part of the arrangement which alone is subjected to great heats, renders it little liable to injury; and together with the cheapness of the materials of which it is constructed, occasions but a very trifling expense for replacing it when injured.

The calculation of the absolute expansion of the bar indicated by the scale may be performed as follows:—As radius to double the sine of half the arc read off, and found in a table of natural sines, so will the radius B be to the chord of the same arc; and this divided by ten (the radius of B being ten times the length of the radius $f\,h$) will give the length required. Suppose the arc read off upon the scale to be 4°,

Radius. Sine of 2°. Inches. Inch.
then 1.0000000 : 0348995 × 2 : : 5 : .3489950 ÷ 10 = .0348995.

Now in working out this proportion it will be observed, that the multiplication by 2 and by 5 being both constant may, in conjunction with the division by 1.0, be omitted; and leaving out also the final division by 10, the case resolves itself into seeking the sine of half the arc, read off upon the scale, in a table of natural sines, and reading it as the decimal of an inch.

Moreover, the chords of small arcs are so nearly proportional to their arcs that, the number of degrees measured upon the scale never exceeding 10, they may be considered without sensible error as denoting equal increments of expansion. The following short Table of the value of a degree, and minutes of a degree, may therefore be useful in practice.

TABLE I.

°	′		Inch.
1	0	=	.00872
0	30	=	.00436
0	20	=	.00290
0	15	=	.00218
0	10	=	.00145
0	5	=	.00072
0	2	=	.00029
0	1	=	.00014

The chord of ten degrees derived from this Table by multiplying .00872 by 10 would therefore be .0872, whereas it is more accurately .0871; but the difference being only $\frac{1}{10,000}$th of an inch may, in most cases, be disregarded.

I shall now proceed to show the degree of confidence which may be placed in this new pyrometer, by comparing the result of its indications with those of the best experiments upon the expansion of metals. Those of MM. Dulong and Petit* are well adapted to this purpose. These able philosophers, in their celebrated prize Memoir on the Measure of Temperatures, and on the Laws of the Communication of Heat, have given, from experiment, the expansion of rods of platinum and iron at different intervals between the freezing point of water and the boiling of mercury. Their mode of experimenting was unexceptionable; but it is to be regretted that they have not corrected their final results for an error of calculation which has been pointed out by Mr. Crichton†, which is by no means unimportant to the reasoning which they have founded upon them. The error, however, affecting the amount of expansion in volume, is reduced to one-third in the linear expansion, which is the subject of the present investigation, and may therefore be disregarded.

The following Table of the expansion of iron and platinum is extracted from their work.

Table II.

Temperature deduced from the dilatation of Air.	Mean absolute dilatation of Iron for 180 degrees.	Mean absolute dilatation of Platinum for 180 degrees.
From 32° to 212°	$\frac{1}{28200}$	$\frac{1}{37700}$
From 360° to 572°	$\frac{1}{22700}$	$\frac{1}{36300}$

Whence we deduce the linear expansion of platinum for 180° Fahrenheit, from 32° to 212° .00088420: and for 180°, from 360° to 572° .00091827: and of iron, from 32° to 212° .00118203: from 360° to 572° .00146842, showing an increasing dilatation in each when referred to an air-thermometer.

The bars of the different metals used in the following experiments were all exactly 6.5 inches in length.

Exp. 1. A square bar of platinum $\frac{3}{10}$ths of an inch thick, was carefully arranged in the black-lead register, which was placed in the apparatus represented, upon

* Ann. de Chimie et Physique, vii. 113. † Annals of Philosophy, New Series, vol. vii. p. 241.

a diminished scale, at fig. 3. *a* is an iron tube about two inches diameter, and closed at the bottom: *b* is a black-lead tube closed at the top, and fitted to the mouth of the former by grinding: *c* is a smaller black-lead tube projecting from the side of the latter near its upper end, and likewise fitted to its place by grinding. The whole forms a kind of alembic, which may be readily put together, and in which mercury may be easily boiled on a common fire, and the vapours collected without loss or annoyance to the operator. The register was fixed in its place by a wire, so that when mercury was poured into the iron bottle it was prevented from floating. The mercury in this experiment rose a little above half the length of the register. The whole apparatus was then placed upon a fire, and in ten minutes the mercury began to boil: in ten minutes more it freely distilled over; and in ten minutes further the apparatus was removed, the register taken out and allowed to cool. The arc measured upon the scale was in this instance 1° 17′.

The experiment was repeated, merely having the head of the alembic off, and suffering the mercury to boil freely in the iron bottle for a quarter of an hour. The arc measured was 1° 23′.

The register was next allowed to float upon the mercury, so that when the head of the alembic was adjusted and the mercury made to boil, it was not immersed in the metal, but surrounded by its vapour: the reading was 1° 16′. A repetition of this arrangement gave 1° 23′.

In another repetition of the experiment, the time was extended to twenty minutes from the first boiling of the mercury; the reading of the scale was 1° 20′.

Again; the time was reduced to ten minutes, and the measurement was 1° 23′.

In the various repetitions of this experiment the mercury freely distilled over, and the temperature was such, that every part of the black-lead tubes, in which the vapour circulated, would just scorch, but not blacken, a piece of writing paper held against them.

The following Table collects these results into one view, and exhibits the expansion denoted by each reading, and the mean result.

TABLE III.

	° ′		
	1 17	=	.01119
	1 23	=	.01206
	1 16	=	.01105
	1 23	=	.01206
	1 20	=	.01163
	1 23	=	.01206
Mean	1 20	=	.01163

The temperature of the atmosphere was about 64° during these observations.

EXP. 2. A bar of soft iron, of the same dimensions as that of platinum, was substituted for the latter in the register. The experiment was repeated five times; twice with the register immersed in the mercury, and three times exposed only to the vapour. The time of exposure varied from twenty minutes to ten, from the first moment when the metal began to boil.

The following Table exhibits the several readings and the appropriate expansions.

TABLE IV.

	° ′		
	2 13	=	.01933
	2 33	=	.02224
	2 10	=	.01890
	2 23	=	.02079
	2 20	=	.02036
Mean	2 20	=	.02036

The greatest variation from the mean was therefore only $\frac{6}{10,000}$dths of an inch in the platinum experiment, and $\frac{13}{10,000}$dths in the iron.

We shall now compare these results with the preceding determinations of MM. DULONG and PETIT.

The expansion of Platinum.

From 32° to 212° = .00088420 × 6.5 (Length of Bar.) . . .	= .005747300
From 360° to 572° = .00091827 × 6.5	= .005968755
	.011716055
From 212° to 360° = Mean of the above . . .	= .005858027
Total expansion from 32° to 572°	= .017574082
Add for the expansion from 572° to 660°, the temperature of boiling mercury, calculated at the highest rate :— 180° : .005968755 : : 88° : .002918058 . .	= .002918058
	.020492140
Deduct expansion for 32°, the experiment with the pyrometer having been made at 64°	= .001021742
Calculated at the lowest rate :— 180° : .005747300 : : 32° : .001021742	
Real expansion of the bar by DULONG and PETIT	= .019470398
If from the real expansion thus obtained	.01947
We deduct the apparent expansion obtained by the pyrometer	.01163
The remainder	.00784

will be the expansion of the black-lead.

The expansion of Iron.

From 32° to 212° = .00118203 × 6.5 (Length of Bar.) . . .	= .007683195
From 360° to 572° = .00146842 × 6.5	= .009544730
	.017227925
From 212° to 360° = Mean of the above . . .	= .008613962
Total expansion from 32° to 572°	= .025841887
Add for the expansion from 572° to 660°, the temperature of boiling mercury, calculated at the highest rate :— 180° : .009544730 : : 88° : .004666311 .	= .004666311
	.030508198
Deduct expansion for 32°, the experiment with the pyrometer having commenced at 64° . . .	= .001365901
Calculated at the lowest rate :— 180° : .007683195 : : 32° : .001365901	
Real expansion of the bar by DULONG and PETIT	.029142297

From the real expansion	.02914
Deduct the apparent expansion obtained by the pyrometer	.02036
The remainder	.00878

is again the expansion of the black-lead as obtained by this series of experiments.

Expansion of 6.5 inches of Black-lead.

From 64° to 660° by platinum bar	.00784
by iron bar	.00878
Mean	.00831

either determination differing from the mean by less than $\frac{5}{10,000}$dths of an inch.

This close agreement in results from two metals whose expansions differ so much from each other is highly satisfactory; but the great delicacy of the instrument may be still better appreciated from the following experiment of the expansion of nine different metals from the temperature of 62° (the temperature of the air at the time of observation) to 212°.

Exp. 3. Bars of the following metals were successively placed in the register and immersed in hot water, which was gradually heated to the boiling point, and kept boiling for ten minutes in each instance. The following Table exhibits the readings of the scale and the appropriate expansions.

Table V.

Platinum . .	0°	19′	=	.00276 from 60° to 212°
Iron (soft) . .	0	35	=	.00508 ———
Copper . . .	0	47	=	.00683 ———
Tin (grain). .	0	56	=	.00814 ———
Zinc	1	40	=	.01454 ———
Lead	1	25	=	.01223 ———
Brass	0	55	=	.00799 ———
Gold (fine) . .	0	36	=	.00552 ———
Silver (fine) .	0	56	=	.00814 ———

In the subsequent Table I have given the absolute expansions of the same metals from 32° to 212° from the best authorities; and for the sake of compa-

rison have added from calculation their expansion from 62° to 212°, by reducing the former in the proportion of 180 : 150.

TABLE VI.

	Length of Bar.	From 32° to 212°.	From 62° to 212°.	Authorities.
Platinum	.00088420 × 6.5	= .005747300	= .004789416	DULONG & PETIT.
Iron	.00118203 × 6.5	= .007683195	= .006402662	DULONG & PETIT.
Copper	.00171821 × 6.5	= .011168365	= .009306970	DULONG & PETIT.
Tin	.00217298 × 6.5	= .014124370	= .011770308	LAVOISIER & LAPLACE.
Zinc	.00294200 × 6.5	= .019123000	= .015935833	SMEATON.
Lead	.00284836 × 6.5	= .018514340	= .015428616	LAVOISIER & LAPLACE.
Brass	.00193000 × 6.5	= .012545000	= .010454166	SMEATON.
Gold	.00146606 × 6.5	= .009529390	= .007941158	LAVOISIER & LAPLACE.
Silver	.00190974 × 6.5	= .012413310	= .010344424	LAVOISIER & LAPLACE.

Upon deducting from the amount of these several absolute enpansions the apparent expansions in the black-lead register, we shall obtain the expansion of the latter from 62° to 212°, as derived from the several metals. The results are comprised in the following Table.

TABLE VII.

	Expansion of the Metal Bars.		Expansion of Black-lead Register.	Difference from Mean.
Platinum	absolute	.00478		
	apparent	.00276		
			=.00202	−.00032
Iron	absolute	.00640		
	apparent	.00508		
			=.00132	−.00102
Copper	absolute	.00930		
	apparent	.00683		
			=.00247	+.00013
Tin	absolute	.01177		
	apparent	.00814		
			=.00363	+.00129
Zinc	absolute	.01593		
	apparent	.01454		
			=.00139	−.00095

TABLE VII. (Continued.)

	Expansion of the Metal Bars.	Expansion of Black-lead Register.	Difference from Mean.
Lead . . .	absolute .01542		
	apparent .01223		
	. .	=.00319	+.00085
Brass . . .	absolute .01045		
	apparent .00799		
	. .	=.00246	+.00012
Gold . . .	absolute .00794		
	apparent .00552	=.00242	+.00008
Silver . . .	absolute .01034		
	apparent .00814		
	. .	=.00220	−.00014
		Mean .00234	

In five instances out of these nine, the difference of the expansion of the black-lead from the mean does not exceed $\frac{30}{100,000}$dths of an inch, two being in deficiency, and three in excess: and it is worthy of observation that they are the metals whose dilatations have always been considered the most regular, and concerning which there is the least difference of authorities, viz. gold, silver, platinum, copper, and brass. The greatest difference is in the tin, which amounts to nearly $\frac{13}{10,000}$dths of an inch in excess; and it is more than probable that the absolute expansion of this metal has not hitherto been obtained with sufficient precision, and that it even varies in different states. I shall return to this subject in the second part of this Paper, which I reserve for a future communication; in which I hope to be able to lay before the Society observations and tables of the dilatations of metals to their melting points. It is my intention in this first part to touch no further upon the subject of expansion than is sufficient to establish confidence in the pyrometer as a measure of heat.

Another confirmation of the precision of these observations may be derived by calculating the expansion of the black-lead register for the 150°, from the greater expansion previously determined by the boiling point of mercury for

596° : .00831 . : : 150° : .00209

which only differs $\frac{25}{100,000}$dths of an inch from the above mean.

Exp. 4. It was a principal object to ascertain whether any and what difference existed in the expansion of different specimens of the black-lead earthenware: two or three registers which I had cut out of the same crucible gave me almost identical results by exposure to boiling mercury. I then selected another specimen by a different manufacturer. Its grain was very fine, and its texture more close and compact than the former. It was twice exposed with the platinum bar to boiling mercury. The first time it was boiled for a quarter of an hour, and the arc measured was 1° 45′. The second time the boiling was continued for only ten minutes, and the reading was precisely the same. The expansion was therefore .01526.

Absolute expansion as before	.01947
Apparent expansion	.01526
Expansion of black-lead	.00421

Exp. 5. The same register of the fine-grained black-lead was exposed for a quarter of an hour with the iron bar to boiling mercury: the arc measured on the scale was 2° 49′ = expansion .02457.

Absolute expansion as before	.02914
Apparent expansion	.02457
Expansion of black-lead	.00457
Fine-grained black-lead by platinum . .	.00421
by iron . . .	.00457
Mean .	.00439

The two experiments differing from the mean by less than $\frac{5}{10,000}$dths of an inch. This shows that the fine-grained ware expands less than the coarser, and proves the necessity of ascertaining the expansion of each register for itself by boiling in mercury; at least till some means be taken to insure their uniform composition. Every register should also be marked with a reference to its proper expansion; and I would recommend all those who may use the instrument for delicate researches, to verify this point for themselves; as they may easily do with the apparatus before described.

Exp. 6. The expansion of the last specimen of black-lead ware being nearly the least which has fallen under my observation, I repeated with it the experiment of the dilatation of six of the former list of metals to the boiling point of water; as the accuracy of these observations is a point which it is of the greatest importance to establish.

The subjoined Table contains the results.

Table VIII.

Platinum . .	0° 22′	= .00319 from 60° to 212°
Iron	0 39	= .00566 ————
Copper . . .	0 54	= .00785 ————
Brass . . .	0 59	= .00857 ————
Gold	0 41	= .00595 ————
Silver . . .	0 58	= .00843 ————

The differences of the observed expansions and the real are also subjoined and ranged by the side of those obtained by the first series of observations.

Table IX.

	Expansion of the Metal Bars.	Expansion of Black-lead. Second Series.	Differ. from Mean.	First Series.	Differ. from Mean.
Platinum	absolute .00478 apparent .00319	=.00159	.00000	.00202	−.00032
Iron . .	absolute .00640 apparent .00566	=.00074	−.00085	.00132	−.00102
Copper .	absolute .00930 apparent .00785	=.00145	−.00014	.00247	+.00013
Brass. .	absolute .01045 apparent .00857	=.00188	+.00029	.00246	+.00012
Gold . .	absolute .00794 apparent .00595	=.00199	+.00040	.00242	+.00008
Silver .	absolute .01034 apparent .00843	=.00191	+.00032	.00220	−.00014
	Mean .	.00159		.00234	

The agreement of this second series with each other is quite as close as that of the first; and it is worthy of remark, that the greatest variation from the mean is in both cases with the iron in deficiency, and nearly to the same amount of one half. It is not unlikely, therefore, that there may be some error in estimating the absolute dilatation of this metal, which is probably something greater than we have assumed.

If we estimate the expansion for these 150° to the boiling point of water from the result obtained by the boiling of mercury, we shall have the following proportion:— 596° : .00439 :: 150° : .00110
which does not differ quite $\frac{5}{10,000}$dths of an inch from the foregoing mean.

Having thus, I trust satisfactorily, established the accuracy of the pyrometer, and the degree to which confidence may be placed in its indications, I shall conclude this part of my subject with the details of some experiments upon the fusing points of different metals. I shall designate the registers of coarse and fine-grained black-lead respectively by the letters A and B.

Exp. 7. About 30lbs. of the clippings of thin sheet copper were very gradually melted in a crucible in the blast furnace of the Royal Institution. The platinum bar was adjusted in the register B, and when the metal was about half run down, it was placed perpendicularly with the index upwards in the crucible, and held down with a pair of tongs. The crucible was then gradually fed with the clippings till the melted metal covered about two-thirds of the register. In this situation it was kept ten minutes, and when it was lifted out some of the metal remained unmelted. A crust of oxide, mixed with metal, had also affixed itself to the upper part of the black-lead. This was partially dissolved away and loosened by immersing the register with great care, when cold, in a diluted mixture of sulphuric and nitric acids. The whole was thus easily removed, and the black-lead exhibited a perfectly clean surface. The arc measured upon the scale was 5° 49′, denoting an expansion of .0508. The temperature of the laboratory was about 65°.

I am indebted to the kindness of Mr. Mathison for unexceptionable opportunities of taking the melting points of gold and silver at the Royal Mint, who also most obligingly assisted me in the operations. Two new registers were prepared, which I shall designate as II and III: their rates of expansion were not determined till after the experiments.

Exp. 8. The register II was carefully adjusted with the platinum bar.

About 90lbs. of fine gold were weighed, and one of the ingots was cut into ten pieces for the purpose of gradually feeding the crucible, and keeping the temperature down to the true melting point during the observation. The remainder was melted in a black-lead crucible in a wind-furnace. When just fused, one of the pieces was thrown in, and the melted metal immediately congealed upon the surface. The register, which had been slowly heated in another crucible to a dull red, was then taken up with a pair of tongs and plunged perpendicularly into the gold about two-thirds of its height. In this situation it was kept ten minutes, and during the time two more lumps of the metal were thrown in. It was then carefully lifted out and set apart to cool. Its surface was perfectly clean, only a few small globules adhering to it, which were easily removed. I may here remark that stirrers of the black-lead earthenware are constantly used at the Mint for agitating the melted gold. The arc measured from this experiment was 6° 10′, equivalent to an expansion of .0537. Temperature of the air about 65°.

Exp. 9. The register III was fitted with the iron bar, and also heated to a dull red. The temperature of the melted gold was prevented from rising by constant feeding with the pieces; the crucible being never left without some portion unmelted. It was then plunged beneath the surface of the metal as in the preceding experiment, and held in that situation for ten minutes. The arc measured was 9° 2′, indicating an expansion of .0787.

Exp. 10. The rates of expansion of the two last registers were determined by boiling them for ten minutes each in mercury. The results were as follow:

	Arc.	Expansion.
II with the platinum bar	1°.50	= .0159
III with the iron bar	2°.38	= .0229

Exp. 11. About 50lbs. of pure silver were melted in a black-lead pot: a little scum floated upon the surface, which appeared at first like drops of oil upon a basin of water. I was afterwards informed that the metal had been refined with nitre, and the dross was owing to the action of a little remaining potash upon the crucible. Two registers had been prepared for the platinum and iron bars; but the observations were lost from the same action upon their substance. They were so deeply corroded in a line which corresponded with the level of the fluid metal, as to render it impossible to apply the scale, with any certainty, to their surfaces.

Exp. 12. Two new registers were selected, whose rate of expansion was

found by boiling in mercury to be equal; the arc in both cases being with the platinum bar 1° 20′ = .0116. They were marked IV and V.

IV was adjusted with the platinum bar. An ingot of silver, which had been refined by cupellation, weighing about 35lbs., was placed in a black-lead crucible in a wind-furnace. When somewhat more than three-fourths were melted, the register, previously heated to a dull red, was plunged into it as before, and held down for ten minutes. When lifted out its surface was found perfectly good, and the few adhering globules of metal were easily removed. When cool the scale was applied, and the arc found to be 4° 10′ = expansion .0363. Temperature of the air 65°.

Exp. 13. The iron bar was placed in the register V, and having been previously heated was plunged into the same pot of metal. The silver at first set about the black-lead and adhered to it in a large lump. At the expiration of ten minutes this was just melted off, and the instrument was raised out of the crucible in a perfectly clean state. When cool the arc measured was 7° 24′ = expansion .0645.

Exp. 14. I made several attempts at the Royal Institution to ascertain the melting point of cast iron; but owing to the large quantity of the metal necessary; to the difficulty of keeping the temperature steady by constant feeding; and to the failure of crucibles,—I did not succeed. I am under obligation to Mr. Parker of Argyle Street, for the readiness with which he afforded me every facility of performing the experiment at his foundry.

I selected a new register for the occasion, which was marked I. Its rate of expansion was not determined till after the experiment. A crucible was prepared capable of containing about 35lbs. of the metal. It was filled with pieces of the best grey iron, and placed in a powerful wind-furnace, which admitted of the operator standing immediately above the crucible with complete command over it. When the metal was melted, the crucible was lifted from the furnace, and the dross skimmed off its surface. It was then replaced; a lump of the same iron was thrown into it, and the register, previously heated red hot, was immersed in the fluid to about the same depth as in the former experiments. It was kept in this situation by means of a pair of tongs for ten minutes, and afterwards gently lifted out and laid upon hot sand. A thin scale of iron adhered to the black lead, which when cold was easily removed,

and retained the form of the bar like a sharp cast, and left the surface of the register perfectly clean and bright. The arc measured after the experiment was 6° 16′ = expansion .0546. Part of the lump of metal remained unmelted.

Exp. 15. Another register, which had been prepared with the iron bar, was immediately immersed in the fluid metal. The fire, however, had been allowed to fall, and the iron almost instantly congealed; and in attempting to lift the register out, it was found to be set fast and broke. The experiment was so far instructive, that it proved how nearly the exact melting point had been attained in the preceding experiment. The iron bar was removed uninjured.

Exp. 16. The register I with the platinum bar was boiled in mercury for ten minutes: the arc afterwards measured was 1° 20′ = expansion .0116.

Exp. 17. About 30lbs. of zinc were carefully melted in a crucible set in a common fire, assisted with the bellows. The register A was prepared with the iron bar and held down in the metal, which was supplied from time to time so as to insure its very gradual fusion, and some portion always remaining in the solid state. In ten minutes time it was removed, and when cold the arc measured was 2° 45′, equivalent to an expansion of .0239.

A dry stick of deal plunged into the melted metal for a few seconds caused a violent ebullition, and was deeply charred. The zinc in this state did not appear red in the light.

Exp. 18. About 12lbs. of zinc were melted in a smaller crucible: the register B prepared with the iron bar was immersed in it; but instead of being gradually supplied, the heat was allowed to increase after fusion till it began to burn: at this point there was an evident blush of red upon its surface. The arc measured upon this occasion was 4° 7′ = expansion .0358.

I shall now collect together the results of the preceding experiments, for the purpose of showing what conclusions may be derived from them with regard to the degrees of temperature which they indicate when referred to the common thermometric scale. I shall make the calculations first upon the supposition that equal amounts of expansion denote equal increments of temperature; and I shall thus be enabled to compare the present series with that which I formerly obtained with my first pyrometer, and to offer a few remarks upon the differences of the two.

I shall adopt the corrected temperature of 662° (350° Centigrade) for the boil-

ing point of mercury, as proposed by MM. Dulong and Petit; which agrees very closely with the amount employed in my first calculations, and which, deducting 62° for the mean temperature at which my experiments commenced, gives 600° for the interval for which the several expansions were determined.

The first column of the following Table refers to the number of the experiment; the second to the mark of the register and the bar which was employed; and the third to the amount of expansion in the same, occasioned by boiling mercury or 600° of temperature upon Fahrenheit's scale. The fourth column exhibits the arc measured upon the scale; and the fifth the equivalent expansion. The sixth contains the corresponding temperature; the seventh records the state of the metal, which was the object of the experiment; and in the eighth I have recapitulated the corresponding results of my former Essay.

Table X.

No. of Experiment.	Mark of Register and Bar.	Expansion for 600°.	Arc measured on Scale.	Expansion.	Temperature.		Metals observed.	Temperature by former Pyrometer.
7	B Platinum	.0152	° ′ 5 49	.0508	2005+65	2070°	Copper, fusing point.	2548°
8	II Platinum	.0159	6 10	.0537	2026+65	2091	Gold, fusing point.	2590
9	III Iron	.0229	9 2	.0787	2061+65	2126	ditto ditto	
12	IV Platinum	.0116	4 10	.0363	1877+65	1942	Silver, fusing point.	2233
13	V Iron	.0203	7 24	.0645	1906+65	1971	ditto ditto	
14	I Platinum	.0116	6 16	.0546	2824+65	2889	Iron, fusing point.	3479
17	A Iron	.0203	2 45	.0239	708+65	773	Zinc, fusing point.	648
18	B Iron	.0245	4 7	.0358	876+65	941	Zinc, inflaming.	

The most remarkable fact displayed by the preceding comparison is the beautiful accordance of the results obtained from two metals whose expansions are so different as those of platinum and iron. The temperature indicated by the latter exceeds that by the former in the instance of the fusing

point of gold 35°, and in that of silver only 29°; and this excess is in accordance with the conclusion of MM. Dulong and Petit, exhibited in Table II., that the expansion of iron increases in the higher degrees in a greater proportion than that of platinum.

The discrepancy between the temperatures derived from the observations with my first pyrometer and the present are considerable, but may be sufficiently accounted for by the differences in the circumstances of the experiments, without imputing inaccuracy to the instrument. In the paper to which I have before alluded, I stated that "I did not offer the results as positive and accurate determinations of the different degrees, but only as nearer approximations than any that had yet been furnished from actual observation. The only method which I had it in my power to adopt for the purpose, I do not consider to be susceptible of absolute accuracy. The arrangement made consisted of a muffle of black-lead placed in an excellent draught-furnace. This muffle was furnished with a door, through a round hole in which the stem of the pyrometer was passed up to the shoulder. Another door, which could be stopped at pleasure, admitted a full view of the interior. The metal to be tried was placed in a small black-lead receptacle, of the same thickness as the pyrometer tube, in the middle of the muffle. Now it is evident that the pyrometer so situated would indicate the mean heat of the whole of the muffle; which heat might, and did, vary in different parts. Of two pieces of silver of the same size placed within an inch of each other, one fused some time before the other." I also suggested that "means might be contrived to surround the instrument with the metal in a state of fusion; but that it required particular opportunities, which it was to be hoped that those would avail themselves of who had them in their power."

That the latter method is the only one which can admit of accuracy will be evident from a few reflections. Setting aside the inequality of the heat of different parts of the same heated muffle, which however is a consideration of the utmost importance, it is obvious that its temperature must considerably exceed the true melting point of the metal exposed to its influence. Just as a piece of ice would never melt in a chamber of the temperature of 32°, but would require a considerably higher heat in proportion to its mass to supply the caloric which becomes latent during the process,—a mass of iron would

exhibit but little signs of liquidity till subjected to a heat much above its true point of fusion. When once in a liquid state, both would rapidly rise to the temperature of the medium to which they were exposed. When metals are melted for the purposes of the arts, they of course require to be heated very far beyond their fusing points, that they may flow into the minutest fissures of the moulds in which they are cast, notwithstanding the cooling influences to which they are suddenly exposed. In some of the finer castings of brass, the perfection of the work depends upon the intensity to which the metal is heated, which in some cases is urged even beyond the melting point of iron. With a fire whose power in all cases must so greatly exceed the temperature required, it is necessary to bestow great care in supplying the metal gradually, as we have before described; as it is inconceivable with what rapidity it rises after the solid pieces are completely dissolved. Evidence of the same fact may be derived from the experiments of MM. Clement and Desormes, which I have before quoted. They calculated the heat of melted iron at 3988°, and of iron just on the point of melting at 3164°,—a difference of 800°. And it is clear from the circumstances of the experiment, that the former must have considerably exceeded the true melting point, or it never could have been transported in a liquid state from the crucible to the apparatus in which the water was heated or the ice melted. It is probable that the process which they employed, of the calorimeter, was not susceptible of great accuracy; but the discrepancy of the results from those which I obtained from the metal in analogous circumstances is not great.

Iron just melting	3164°	by the former
	2889	by Pyrometer
	——	275° difference.
Iron melted at a high heat .	3988	by the former
	3479	by Pyrometer
	——	509° difference.

A similar excess also appears in their determination of the heat of melted copper, and obviously admits of the same explanation.

After performing these experiments upon the melting points of the metals, I was desirous of ascertaining the effects of the most intense heat which it was possible to produce in a furnace; and to measure the utmost limits of expansion

in a platinum bar. For this purpose I made use of an excellent wind-furnace in the Royal Institution, in which upon former occasions hob-nails had been completely fused into a button.

Exp. 19. The register I, which had not been the least injured by the previous experiments, was fitted with a new bar of platinum, which had been drawn as a wire, was $\frac{5}{8}$ths of an inch in diameter and very ductile. The iron bar was also adjusted to a new register, and both were placed upright in a well heated crucible. About half an inch of powdered charcoal was strewed upon the bottom to prevent any adhesion; and two soft iron nails, and a piece of unglazed Wedgwood's porcelain, were thrown in for the purpose of affording some indication of the degree of heat attained. The crucible was then set in the furnace, another smaller crucible inverted upon it, covered with coke, and the heat urged to the utmost for two hours. The fire was suffered to burn out, and the crucible with its contents removed for examination. It was sound, but the luting had been completely fused. The nails were found melted into two complete buttons, and the porcelain was partially fused upon the surface.

The register I appeared to be uninjured, but the platinum ring and wedge were loose, evidently from a contraction having taken place in the substance of the black-lead. This was no doubt owing to the heat having exceeded that at which it had been originally baked. The amount of expansion consequently could not be measured. The platinum ring, both of this and the other register, exhibited a remarkable change of texture; they had become very rough and crystalline, and were perfectly brittle, breaking easily between the fingers. The platinum bar also, which there was some difficulty in removing from the cavity, presented a very extraordinary appearance. It was apparently embossed with crystals, and was evidently larger at the lower end than at the top: it was also something contracted in length. Upon examination with a lens no regular facets could be detected, but it had the appearance of a bar constructed of plates of native platinum loosely welded together.

The register which contained the iron bar was considerably bent, and had several transverse clefts in its substance, owing possibly to its having become inclined in the crucible. Partial fusion had taken place upon the surface of the bar, which had run down and formed a knot at its lower extremity. About

an inch of the same end was found to have been converted into steel, but all the rest retained the character of soft iron.

Exp. 20. I repeated the last experiment with the same platinum bar in the register I. The arrangement was precisely the same, with the exception of the second register with the iron bar, and the fire was maintained with equal intensity for an equal time.

The iron nails were found perfectly melted, and the porcelain superficially fused as before. The ring and wedge, however, were fixed in their places, and the index undisturbed, but the measure was unfortunately lost from an accident. The texture of the platinum ring was changed, as in the previous experiment, and the bar tightly fixed in the cavity. By frequent gentle concussions it was removed without injury to the black-lead, which had some slight marks of fusion upon its surface, but was in a perfectly good condition. The bar was in a still rougher state than before, highly crystalline, and exhibited several large longitudinal clefts in its substance. It was found, by measurement with callipers, to be $\frac{1}{50}$th of an inch larger in diameter at its lower than at its upper end, and seemed to be approaching a state of complete disintegration. It was, however, perfectly hard and inflexible. My intention was to have again exposed it for several hours to the same degree of heat, with the expectation that the disintegration would have been complete, and that it would actually have fallen in pieces during the operation: in the mean time I chanced to make it red hot upon a common charcoal fire; and upon attempting to lay hold of it with a pair of tongs the two ends dropped off, and I only withdrew the small portion which I had grasped, and which was flattened and fractured by even this slight compression. The two ends were afterwards carefully, but with difficulty, raised from the fire, and when cold were perfectly hard and inflexible. I again heated a portion of the bar to a dull red, and it crumbled to powder from a slight blow with a hammer.

Exp. 21. It being a point of the greatest interest to ascertain the maximum of expansion which took place in the platinum previous to this remarkable change of structure, I adjusted the original platinum bar, with which the greater part of my experiments had been made, and which presented a perfectly smooth surface, and was very soft and ductile, in the register I. A crucible was placed in the same wind-furnace, containing only a little charcoal

powder, with the iron nail and fragments of porcelain as test pieces. The fire was urged to the utmost; and when it had been continued two hours the cover was removed, and the register, previously made red hot, was carefully introduced, the cover replaced, and the ignited fuel heaped upon it. At the expiration of a quarter of an hour it was lifted out and cautiously cooled. An excellent measure was obtained, and the arc determined to be 7° 24′ = expansion .0645.

The test pieces were found in the same state as in the previous experiments. The platinum bar was loose in the cavity, and had not altered its form; but its surface had assumed a slightly crystalline texture, and it had become very hard and inflexible.

The expansion registered would, upon the hypotheses before assumed of equal amounts of expansion denoting equal increments of temperature, indicate a heat of 3336°; or, adding the initial temperature 65°, = 3401°. But it must be remembered that this is probably rather the temperature at which the change in the structure of the platinum took place, than the utmost heat of the furnace. The latter may possibly exceed the degree at which the expansion of the metal ceases, and at which its particles evidently form a new arrangement; but this point cannot at present be determined. The coincidence of this result with that obtained in the former series of my experiments, is very remarkable. The temperature at which I obtained the fusion of cast iron at that time was calculated at 3479°, and was produced by the utmost energy of an excellent wind-furnace; and this, it will be observed, is within 80° of the present maximum.

Exp. 22. Being desirous of ascertaining whether the register and platinum bar had undergone any change in their rates of expansion by the intense heat to which they had been exposed, I again adjusted the latter in the register I, which had now been once immersed in melted iron, and three times subjected to the action of the wind-furnace, and boiled them for ten minutes in mercury. The arc measured was 1° 19′ = expansion .01148: the difference of 1′ may safely be ascribed to the uncertainty of the reading.

The temperatures thus determined will require correction, if we adopt the conclusion derived from the experiments of MM. Dulong and Petit,—that the dilatability of solids, referred to an air-thermometer, increases with the heat.

The amount of this correction will be as the rate of increase; and according to those gentlemen is 11°.6 of the Centigrade thermometer, or 20°.8 Fahr. from 32° to 572°, or the calculated temperature is to the true as .00091827 : .00088420. Supposing the increase of dilatability to continue the same for equal intervals of temperature, which however has not yet been proved, the following Table will exhibit the corrected temperatures derived from the preceding experiments with the platinum bar.

Table XI.

	Observed.	Corrected.
Melting point of Silver	1942	1873*
——— Copper	2070	1996
——— Gold	2091	2016
——— Iron	2889	2786
Temperature of the maximum of expansion of Platinum	3401	3280

If we reason in the same way from the increase of the dilatation of iron, as laid down by the same authors, the discrepancy between the temperature derived from the platinum and iron is very considerable; the melting point of silver coming out 1682°, and that of gold 1815° by the latter: but I conceive that the determination of this point in the iron is open to objections which do not apply to the platinum, and my suspicion is confirmed by the anomalous expansion of the iron exhibited in Tables V. and IX., and to which I shall recur upon a future occasion.

The general utility of the pyrometer, however, will in no way be affected by any uncertainty in these corrections. The indications which it is capable of affording will always be positive determinations, which it will be easy to modify by calculation, as our theories may improve. For all common purposes (and I must own that I look forward with hope that this instrument will prove eminently useful in many of the common processes of the arts) it will not even be necessary to note the expansion indicated by the arc measured; but each minute of the degree may at once be valued in degrees of Fahrenheit's scale at the time of taking its rate of expansion by the boiling of mercury: and a

* Mr. Prinsep, from a laborious series of experiments upon the expansion of air confined in a bulb of gold, determines the melting point of silver to be 1830°.—Phil. Trans. 1828. p. 94.

Table of such values should be furnished for each register by the maker of the instrument. The following, for example, would be the proper Table for register I, which has been so often referred to, in which the arc for the boiling of mercury or 600° (without adding the initial temperature) was 1° 20′.

TABLE XII.

			Expansion.		Temperature.
1°	0′	=	.00872	=	450°
0	30	=	.00436	=	225
0	20	=	.00290	=	150
0	15	=	.00218	=	112
0	10	=	.00145	=	75
0	5	=	.00072	=	37
0	2	=	.00029	=	15
0	1	=	.00014	=	7.5

With such a Table an intelligent workman could employ the instrument without any material error. Those who might object to the expense of a platinum bar may substitute an iron one for ordinary purposes, and the cost of the black-lead register can never be an obstacle to its general use. Other substances might obviously be employed in its construction, but the facility with which it can be worked, its small expansion, its infusibility, and the impunity with which it bears the most sudden changes of temperature (as when red hot it may even be quenched in water without injury), will probably always give the black-lead ware the preference. The only precaution to be taken with it is to expose it previously, out of the contact of air, to a heat at least as great as that in which it is intended to employ the instrument.

XXI. *On the Phenomena and Laws of Elliptic Polarization, as exhibited in the Action of Metals upon Light.* *By* David Brewster, *LL.D. F.R.S. Lond. & Edin.*

Read April 22, 1830.

FROM the first dawn of the science of polarization, the action of metals upon light has presented a troublesome anomaly. Malus at first announced that they produced no effect whatever; but by employing a different method of observation, I found that the light reflected by metallic surfaces was so far modified as to produce, when transmitted through thin crystallized plates, the complementary colours of polarized light. From a second series of experiments made previous to mine, Malus came to the conclusion, that the difference between transparent and metallic bodies consisted in this: that the former refract all the light which they polarize in one plane, and reflect all the light which they polarize in another; while metallic bodies reflect what they polarize in both planes.

Having discovered the property of transparent bodies to polarize light by successive reflexions at angles at which a single reflexion produced no perceptible effect*, I resolved to apply this method of examination to metals; and on the 7th of February 1815, when I first made the experiment, I discovered the curious property possessed by silver and gold of dividing a polarized ray into complementary colours by successive reflexions. As this subject promised to open a wide field of inquiry, I prepared for the ardent prosecution of it with all the metallic bodies which could be procured; but the pressure of professional business prevented me for about a month from doing any thing very effectual.

On the 6th of March 1815, I received a letter from M. Biot, requesting some information on a matter of business; and in answering this letter on the same day, I communicated to him an account of the discovery above men-

* Phil. Trans. 1815, p. 142.

tioned*. Immediately after this I received the most perfect plates of silver, one pair polished by friction, and another by hammering; two pair of plates of gold, one of jewellers', and another of fine gold; with plates of steel, platinum, palladium, copper, brass, and speculum metal; and with their help I obtained the general result, that a single reflexion from a metallic surface produces the same effect upon polarized light as a certain thickness of a crystallized body, with many other results, which it is unnecessary here to indicate.

As soon as M. Biot had received notice of my discovery, he seems to have devoted himself to the same inquiry; and with all the leisure of an Academician, and the splendid apparatus presented to him by the Institute, he obtained many of the results at which I had arrived, and others to which I have no claim; and on the 29th of March he transmitted to me, through Dr. Wollaston, an open letter containing an abstract of his experiments, and expressing the hope that they would be of use to me in my researches.

Although this expression led me to believe that I should enjoy the privilege of publishing the first account of my own discovery, yet I took the precaution of having all my papers on the subject signed by the Treasurer of the Royal Society of Edinburgh, and I proceeded with new zeal in the further examination of the subject. I soon learned, however, from M. Biot, that he meant to treat the subject in his Traité de Physique; and though I remonstrated against this as a breach of courtesy, I had the mortification to see the discovery, to which I perhaps attached too much importance, published for the first time in a foreign work.

I trust the Society will excuse these details as a necessary apology for having so long delayed to fulfil the promise, more than once made in their Transactions, to communicate to them an account of these experiments†. The

* It is related in the History of Optics, Edinburgh Encyclopædia, vol. xv. p. 493, note, that I communicated this discovery to M. Biot on the day on which it was made:—this is a mistake, as it was done a month afterwards.

† In a letter to Sir Joseph Banks, dated July 28th, 1815, I communicated an abstract of these and other experiments, with a request that he would permit the MS. to remain in his possession, as an evidence of my claims. Sir Joseph complied with this request: but nearly two years afterwards, happening to see the MS., he thought that it had been intended for publication, and laid it before the Royal Society without my knowledge. It was accordingly read on the 23rd of January 1817, under the title of Abstract of Experiments on Light, and ordered to be printed. When the proof-sheet was sent me for correction, I requested the paper to be cancelled, as it was not intended for publication.

reasons which I have assigned were subsequently strengthened by new inquiries which at first threw great doubts over the views which M. Biot and I had taken of the subject, and finally convinced me of the rashness of our generalizations. The study of M. Fresnel's fine discoveries respecting circular polarization enabled me to advance still further in the inquiry; and having more recently resumed the investigation, I trust I shall now be able to present to the Society a satisfactory analysis of the singular phenomena exhibited in the action of metals upon light.

Sect. I. *On the action of metals upon common light.*

When we analyse with a rhomb of calcareous spar a ray of common light, reflected at different angles from a metallic surface, there will be observed in one of the images a defalcation of light, as if a portion of the incident ray was polarized in the plane of reflexion. This effect will be still more distinctly seen if we examine the system of polarized rings formed round the axes of crystals by means of the light reflected from metals. If the light had suffered no modification by reflexion, or if the metal reflected in equal quantities the light polarized in opposite planes, the rings would not be visible at all; but it will be found that they are easily seen in the light reflected by all metals. They are most distinctly visible at an incidence of about 74°, at an average, and become fainter and fainter as the incidence exceeds or falls below that angle. They appear best defined in light reflected from galæna and metallic lead, and with least distinctness in light reflected from silver and gold, as shown in the following Table, in which the metals are arranged in the order in which they exhibit the rings most brightly, and consequently in the order in which they polarize the greatest quantity of light in the plane of reflexion.

Galæna,	Antimony,	Bismuth,	Grain tin,
Lead,	Steel,	Mercury,	Jewellers' gold,
Gray cobalt,	Zinc,	Copper,	Fine gold,
Arsenical cobalt,	Speculum metal,	Tin plate,	Common silver,
Iron pyrites,	Platinum,	Brass,	Pure silver.

If we now take two plates of each of these metals and examine the light which has undergone more than one reflexion, we shall find that the quantity of light which each polarizes in the plane of reflexion increases with every reflexion, and that in several of them the whole incident pencil is completely polarized.

When the luminous object is a wax-candle placed at the distance of ten feet, eight reflexions from a plate of steel at angles between 60° and 80° polarize the whole of the light, while at angles above 80° and below 60° a greater number of reflexions is required. With galæna, lead, cobalt, and antimony, a much smaller number of reflexions polarizes the whole pencil; whereas with pure and highly polished silver a very great number is necessary: the light reflected from the silver becomes redder and redder, indicating an increasing absorption or dispersion of the less refrangible rays.

By the use of common light it would be in vain to attempt to discover the law according to which the polarization of the incident pencil is effected in different metals; but by another mode of analysis we shall be led to the mathematical law for computing the exact proportion of the reflected pencil which is polarized at certain angles when the number of reflexions exceeds one.

Sect. II. *On the action of metals upon polarized light.*

If a pencil of polarized light is received on a polished metallic surface placed so as to have a rotatory motion round the polarized ray, the reflected light will receive no modification (excepting what arises from its property of apparently polarizing a portion of light in the plane of reflexion) when the plane of incidence is inclined 0°, 90°, 180°, and 270° to the plane of primitive polarization; but in every other azimuth of the plane of incidence the reflected pencil will be found to have suffered a remarkable change, which gradually increases as the azimuth of that plane varies from 0° to 45°, from 90° to 135°, from 180° to 225°, and from 270° to 315°. At the azimuths of 45°, 135°, 225°, and 315°, the effect is a maximum, and it gradually diminishes from 45° to 90°, from 135° to 180°, from 225° to 270°, and from 315° to 360°.

In order to investigate the nature of this change, we shall suppose the plane of reflexion from the metal to be inclined −45°, or to the left of the plane of

primitive polarization. In this position let a plate of highly polished steel receive the polarized ray of ordinary intensity. At 89°, 88°, and 87° of incidence, almost no change is produced upon it by the action of the metal. We can easily see that the plane of polarization of the ray is turned from right to left, exactly as it would be by a transparent surface. In like manner at all angles of incidence from 0° to about 40° no decided effect is produced, except the change in the plane of polarization. At angles less than 87° the change begins to appear, reaches its maximum at about 75°, and diminishes gradually to 40°. By means of the analysing rhomb, it is easily seen that a great portion of the original pencil has had its plane of polarization changed from +45° to 0°, as the incidence diminishes from 75° to 0°. If, indeed, we measure the rotation of the principal section of the rhomb when the extraordinary pencil is a minimum at different angles of incidence, we shall find it to correspond with $45° - \varphi$, φ being calculated from the formula $\tan \varphi = \frac{\cos(i + i')}{\cos(i - i')}$ in which $\frac{\sin i}{\sin i'} = 3.732$, the index of refraction for steel. The value of φ will be found to be nearly the same at 87° and 40°, which shows why at these two angles the change under our consideration is just beginning to appear with light of ordinary intensity.

The physical effect of the metallic surface being a maximum at 75°, we shall now examine the character of the pencil reflected at that angle.

1. The pencil thus reflected is not polarized light, because it does not vanish during the revolution of the analysing rhomb.

2. It is not common light, because when we reflect it a second time at 75° from another steel surface, it is restored to light polarized in one plane.

In order to discover its nature, let it be transmitted along the axis of calcareous spar. The system of rings is changed into the form shown in Plate X. fig. 1. almost exactly in the same manner as if a thin film of a crystallized body which polarizes the pale blue of the first order had crossed the system. If we substitute for the calcareous spar films of sulphate of lime, which give different tints, we shall find that these tints are increased according as the metallic action coincides with, or opposes that of the crystal.

On the authority of this experiment I was led to believe that metals acted upon light like crystallized plates; and when I found that the colours were not only better developed, but more pure after successive reflexions, it was a natural though a rash generalization, to conclude as I did, and as M. Biot did after me, that each successive reflexion corresponded to an additional thickness of the crystallized film.

In order to show the incorrectness of this deduction, let a ray polarized +45° be reflected twice from steel at angles of 75°. In this case the effect of the second reflexion should be to double the tint produced by the first, if the tints are those of crystallized plates. The result, however, is, that the whole of the light is polarized in one plane, in place of consisting of two pencils polarized in opposite planes. M. Biot got over this embarrassment by regarding the tint produced by two reflexions as the white of the first order, which, in consequence of its complementary tint being black, is the only one where the light is all polarized in one plane: but had he examined the light reflected four times, six times, or eight times at 75°, he would have still found it all polarized in one plane, a result entirely incompatible with the supposition of the tints rising with the number of reflexions. That the tint is not the white of the first order may be more easily proved by making it pass along the axis of calcareous spar; for we shall find that in place of producing an increment of tint, the effect of the second reflexion has been to destroy entirely the effect of the first, and to restore the ray to common polarized light. All this will appear by the perfection of the system of rings seen through the spar. If we examine in a similar manner the light which has undergone any number of reflexions between the plates, we shall easily ascertain that the effect never exceeds that of a quarter of a tint in Newton's scale.

Having thus ascertained that light polarized + 45°, and reflected at the maximum polarizing angle of metals, is neither common light nor polarized light, nor light constituted like that which passes through thin crystallized plates, I conceived the idea of its resembling circularly polarized light—that remarkable species of light which comports itself as if it revolved with a circular motion during its transmission through particular media.

According to Fresnel's beautiful discovery, a ray of light polarized +45° is

circularly polarized when it has suffered two total reflexions from glass at an angle of $54\frac{1}{2}°$; and when such a ray is made to suffer other two reflexions at the same angle, it is restored to the state of light polarized $-45°$ to the plane of reflexion, whatever be the azimuth of the second plane of reflexion in relation to the first. In like manner I shall proceed to show that a ray of light polarized $+45°$, and reflected once at the maximum polarizing angle from metals and certain metallic ores, has an analogous polarization, viz. a polarization hitherto unrecognized, and intermediate between circular and rectilineal polarization.

Let the ray polarized $+45°$ be reflected at $75°$ from steel, and let a second plate of steel be made to turn round the ray thus reflected. At the azimuths of $45°$, $135°$, $225°$, and $315°$, with the plane of primitive polarization, that is, when the planes of the two reflexions are either coincident or rectangular, the first reflected ray will be restored to polarized light at an incidence of $75°$. At azimuths of $0°$ and $180°$ the restoration will be effected at an incidence of $80°$, while at azimuths of $90°$ and $270°$ it will take place at an incidence of $70°$, and at intermediate azimuths it will take place at intermediate incidences. Hence the ray of light reflected from steel, though it has the general properties of a circularly polarized ray, differs from it in this remarkable particular, that it requires different angles of incidence in different azimuths to restore the polarized light.

In circular polarization, as we have seen, the ray has the same properties in all its sides; and the angles of reflexion at which it is restored to polarized light in different azimuths are all equal, like the radii of a circle described round the ray. Hence, without any theoretical reference, the term circular polarization is from this and other facts experimentally appropriate. In like manner, without referring to the theoretical existence of elliptic vibrations produced by the interference of two rectilineal vibrations of unequal amplitudes, we may give to the new phenomena the name of elliptic polarization, because the angles of reflexion at which this kind of light is restored to polarized light may be represented by the variable radius of an ellipse.

In circular polarization the restored ray has its plane of polarization always inclined $-45°$ to the plane of the second system of reflexions. In elliptic polarization the difference is remarkable. The inclination of the plane of the

restored pencil is likewise —, but always less than 45°, as will appear from the following Table, which contains the greater number of metallic bodies.

Names of Metals.	Angles of Restoration.	Names of Metals.	Angles of Restoration.
	° ′		° ′
Total reflexions .	45 0	Bismuth	21 0
Pure silver . .	39 48	Speculum metal . .	21 0
Common silver .	36 0	Zinc	19 10
Fine gold . . .	35 0	Steel	17 0
Jewellers' gold .	33 0	Iron pyrites . . .	14 0
Grain tin . . .	33 0	Antimony	16 15
Brass	32 0	Arsenical cobalt . .	13 0
Tin plate . . .	31 0	Cobalt	12 30
Copper	29 0	Lead	11 0
Mercury . . .	26 0	Galæna	2 0
Platina	22 0	Specular iron . .	0 0

The bodies in this Table are obviously in the inverse order according to which they polarize most light in the plane of reflexion.

I have inserted at the top of the Table the inclination of the restored pencil in total reflexions, which is 45°; and at the bottom, that of specular iron, which is 0°; in order to show the transition from elliptic polarization to circular polarization on the one hand, and to rectilineal polarization on the other.

In these experiments the primitive ray was polarized +45° to the plane of reflexion; but when this angle diminishes, the plane of the restored ray approaches to the plane of reflexion, and ultimately coincides with it at 0°; and when this angle increases, the plane of the restored ray recedes from the plane of reflexion, and the two planes form an angle of 180° when the other angle becomes 90°.

The following experiments were made with plates of pure silver, in which the inclination φ was 39° 48′, when the inclination x of the plane of polarization was 45°.

Inclination x of the Plane of primitive Polarization to Plane of Reflexion.	Observed Inclination of the restored Ray to the Plane of Reflexion or φ.	Inclination φ calculated by the Formula.
$+90$	$-90\ 0$	$-90\ 0$
85	84 36	84 0
75	74 10	72 10
65	63 51	60 46
55	52 18	49 57
45 $\theta =$	39 48	39 48
35	32 23	30 28
25	23 10	21 14
15	13 16	12 35
5	4 40	4 10
0	0 0	0 0

Calling θ the inclination or value of φ at 45°, we may represent these observations by the formula, $\tan \varphi = \tan \theta \tan x$, and the actual change of the plane of polarization, or R, will be $R = x + \varphi$.

When φ is given, $\tan x = \frac{\tan \varphi}{\tan \theta}$, and when $\varphi = 45°$, and consequently $\tan \varphi = 1$, we have, $\cot x = \tan \theta$, and $x = 90° - \theta$.

Since light polarized $+45°$ is elliptically polarized by one reflexion from steel at 75°, and is restored to light polarized $-17°$ by a second reflexion at 75°, it is clear that a third reflexion at 75° will again polarize it elliptically, while a fourth reflexion at 75° will again restore it to light polarized $+\varphi$, φ being a quantity less than 17°, and given by the preceding formula. The same effects will be reproduced with different numbers of reflexions, as in the following Table.

No. of Reflexions from Steel at 75° of Incidence.	State of the Light Reflected.	Inclination of the Plane of Polarization. Observed.	Calculated.
1	Elliptically polarized.	° ′	°
2	Restored to light polarized	$-17\ 0$	-17

No. of Reflexions from Steel at 75° of Incidence.	State of the Light Reflected.	Inclination of the Plane of Polarization. Observed.	Calculated.
3 . .	Elliptically polarized.		
4 . .	Restored to light polarized . .	$+5^\circ\ 10'$ . .	$+5^\circ\ 22'$
5 . .	Elliptically polarized.		
6 . .	Restored to light polarized . .	$-2\ 0$ . .	$-1\ 38$
7 . .	Elliptically polarized.		
8 . .	Restored to light polarized . .	$0\ 0$ . .	$+0\ 30$
9 . .	Elliptically polarized.		
10 . .	Restored to light polarized . . .	$0\ 0$ . .	$-0\ 9$
11 . .	Elliptically polarized.		
12 . .	Restored to light polarized . . .	$0\ 0$ . .	$+0\ 3$

Hence it follows, that at every odd number of reflexions at the maximum polarizing angle the light is elliptically polarized, and at every even number it is restored to a single plane of polarization. In circular polarization the inclination φ of this plane is always $\mp 45°$, even after fifty reflexions, as I have ascertained by direct experiment; but in elliptical polarization the inclination diminishes at every restoration; and in the case of steel it is reduced to near 0° after eight reflexions, when the light is all polarized in the plane of reflexion; that is, the elliptic polarization gradually diminishes and terminates in rectilineal polarization.

The value of φ, as given in the preceding Table, and consequently the number of reflexions when it approaches to 0°, may be deduced from the formula,

$$\tan \varphi = \tan \theta \,.\, \tan x.$$

After the first reflexion $x = +45°$, and φ, or the inclination of the plane of the ray as restored by the second reflexion, is $= -17°$, as given by experiment. Hence the light which suffers the third reflexion, and is thereby elliptically polarized, is not, as originally, polarized $+45°$, but only $-17°$; and consequently, when it is restored after the fourth reflexion, the value of φ must be such as corresponds to an equality in the values of x and θ, both of them being $= 17°$. Hence the formula becomes,

$$\tan \varphi = \tan^2 x, \text{ or } \tan \varphi = \tan^n x;$$

n being the number of pairs of reflexions, or half the number of reflexions which the restored ray has undergone. In this way the last column of the pre-

ceding Table has been calculated. The same formula represents also, as it should do, the phenomena at the limits of elliptic polarization. In the case of circular polarization, where the plane of polarization of the restored ray is 45°, we have,

$$x = 45°, \tan x = 1, \text{ and } \tan \varphi = \tan^n x = 1, \text{ or } \varphi = 45°$$

after any number of reflexions however great. In like manner, in rectilineal polarization, where $x = 0°$, we have $\varphi = 0°$, that is, the ray is polarized in the plane of reflexion.

The above formula is suited to any series of reflexions at any angle when the value of φ for the first term of the series is known. The value of φ for two reflexions, the first term of the principal series, can be determined only by experiment, and has been given in a former Table for several metals; but we may determine from it the value of φ for the first term of any other series, provided it is an even number, in the following manner. Making $x =$ the inclination for two reflexions at the maximum polarizing angle, and φ the value of x at any number of reflexions $2n$, we shall have,

$$\tan \varphi = \frac{\tan x + \tan^n x}{2}; \qquad (A)$$

where $\tan^n x$ is the value of φ at the maximum polarizing angle for $2n$ reflexions; but as no odd number can occur in the principal series, the preceding rule will not apply to such numbers.

The following Table shows the coincidence between the formula and experiment.

SILVER.

Number of Reflexions.	Values of n.	Angle of Incidence.	Inclination of Plane of Polarization. Observed.	Calculated.
2	1	73° 0′	39° 48′	39° 48′
4	2	82 30	37 45	37 22
6	3	85 6	35 0	35 22

STEEL.

2	1	75 0	17 0	17 0
4	2	83 30	11 30	11 17
6	3	85 45	9 30	9 30

When the number of reflexions which begin the series is odd or fractional, we must determine, by the preceding formula, the value of φ for the even number immediately above it: and calling ν the number of odd or fractional reflexions, and N the number of even reflexions immediately above ν, φ the inclination for N reflexions as given by the formula (A), and φ' the inclination required, we shall have,

$$\tan \varphi' = \tan x - (\nu - 2\left(\frac{\tan x - \tan \varphi}{N - 2}\right). \qquad \text{(B)}$$

The truth of this formula will appear from the following Table.

SILVER.

Number of Reflexions.	Angles of Incidence.	Inclination of the Plane of Polarization. Observed.	Calculated.
3	79° 40′	38° 28′	38° 33′
5	77 13	33 10	33 36
5	84 5	26 0	26 24

STEEL.

3	77 37	13 15	14 11
5	84 38	10 30	10 23

The same results will be obtained at the angles of equal phase below the maximum polarizing angle.

This last rule is suited to even as well to odd numbers of reflexions, but it does not give precisely the same results for even numbers as the formula (A). The difference, however, is far within the limits of the errors of observation. The inclination, for example, at 4 reflexions, is by formula (A) 37° 22′ for silver, whereas by formula (B) it is 37° 34′, the difference being only 12 minutes.

In circular polarization, therefore, the plane of polarization of the restored light continues by successive reflexions to oscillate on each side of the plane of reflexion with a never varying amplitude from $+45°$ to $-45°$; while in elliptical polarization, the same plane oscillates with an amplitude continually diminishing till it is brought to nothing in the plane of reflexion.

In steel, as we have seen, the polarization is highly elliptical, and the amplitude of the oscillations of the plane of restoration is quickly brought to zero;

but in silver, where the polarization approaches nearly to circular, the oscillations diminish very slowly in amplitude, as the following Table shows.

No. of Reflexions from Silver at 75° of Incidence.	State of the Reflected Light.	Inclination of the Plane of Polarization, or ϕ. Observed.	Calculated.
1	Elliptically polarized.		
2	Restored to light polarized	$-38^\circ\ 15'$	$-38^\circ\ 15'$
3	Elliptically polarized.		
4	Restored to light polarized	$+31\ 15$	$+31\ 52$
5	Elliptically polarized.		
6	Restored to light polarized	$-26\ 0$	$-26\ 6$
12	Restored to light polarized		$+13\ 30$
18	Restored to light polarized		$-\ 6\ 42$
36	Restored to light polarized		$+\ 0\ 47$

Owing to the high dispersive power of silver, I found it difficult to carry the comparison any further with white light, as the colours closed in upon the points of evanescence, and rendered it impossible to determine with any precision the inclination of the plane of polarization.

The preceding results afford the clearest explanation of the phenomena which steel and silver exhibit in the reflexion of common light. As common light is similar to two equal pencils polarized $+45°$ and $-45°$, and as steel brings two such pencils into a state of parallelism with the plane of reflexion, common light must therefore be wholly polarized in the plane of reflexion after 8 reflexions. In like manner we see why the same effect is not produced by silver, because after 8 reflexions the two planes of the pencils are inclined 42°, so as to form a partially polarized pencil.

The same results also furnish us with a method of computing the proportion of polarized light in any pencil of common light, reflected from metals at angles at which the restoration of the elliptical polarized pencil is effected. In order to determine this proportion for steel after two reflexions at 75°, we must consider that a pencil polarized $+45°$ is restored by these two reflexions to light polarized $-17°$, and consequently a pencil polarized $-45°$ to light polarized $+17°$. Hence a beam of common light will consist after two reflexions of two pencils $+17°$ and $-17°$ of equal intensity, and consequently in the same

state of partial polarization as if common light had been reflected either at an angle of 45° or 68° from a surface of glass. Consequently in the formula *

$$Q = 1 - 2 \sin^2 \varphi, \quad \text{we have } \varphi = 17° \text{ and } Q = 0.829.$$

Hitherto we have considered elliptical polarization as produced only at the maximum polarizing angle. It may be produced, however, by a sufficient number of reflexions at any given angle either above or below the maximum polarizing angle, as appears from the following Table, in which the reflexions are made from two parallel plates of steel.

No. of Reflexions from Steel at which Elliptic Polarization is produced.	No. of Reflexions at which the Pencil is restored to a single Plane.	Angles of Incidence.	
		Calculated.	Observed.
		° ′	° ′
3, 9, 15, &c.	6, 12, 18, &c.	85 45	86 0
$2\frac{1}{2}$, $7\frac{1}{2}$, $12\frac{1}{2}$, &c.	5, 10, 15, &c.	84 38	84 0
2, 6, 10, &c.	4, 8, 12, &c.	83 30	82 20
$1\frac{1}{2}$, $4\frac{1}{2}$, $7\frac{1}{2}$, &c.	3, 6, 9, &c.	79 39	79 0
1, 3, 5, &c.	2, 4, 6, &c.	75 0	75 0
$1\frac{1}{2}$, $4\frac{1}{2}$, $7\frac{1}{2}$, &c.	3, 6, 9, &c.	68 53	67 40
2, 6, 10, &c.	4, 8, 12, &c.	60 2	60 20
$2\frac{1}{2}$, $7\frac{1}{2}$, $12\frac{1}{2}$, &c.	5, 10, 15, &c.	56 5	56 25
3, 9, 15, &c.	6, 12, 18, &c.	51 24	52 20

The numbers given in the third column are calculated by the following method. The relation of the preceding phenomena to the angle of maximum polarization is obvious; and if we consider the nature of the formula, $\tan \varphi = \frac{\cos (i + i')}{\cos (i - i')}$, we shall see that the angles at which the rectilineal polarization of the primitive pencil is destroyed have a reference to the rotation which the reflecting surface produces in the plane of polarization. The angles indeed in the third column, at which similar effects are produced above and below 75°, are those at which φ has equal values. This is a very important relation, and enables us to determine the phase P of the two inequal portions of oppositely polarized light, by the interference of which the elliptic polarization is produced. It may be expressed by $P = 2R$.

But $R = 45° - \varphi$,

Hence $P = 90° - 2\varphi$,

$$\tan \varphi = \frac{\cos (i + i')}{\cos (i - i')}.$$

* See my Paper "On the Law of the Partial Polarization of Light by Reflexion," *supra*, p. 76.

In this manner we obtain the following results.

No. of Reflexions for Elliptic Polarisation.	Angle of Incidence on Silver.	Angle of Incidence on Steel.	Inclination of Plane, or ϕ.	Rotation of Plane, or R.	Phase, or P.
3	85° 6′	85° 45′	30° 0′	15° 0′	30° = + of 90°
$2\frac{1}{2}$	83 49	84 38	26 15	18 45	$37\frac{1}{2}$ = $\frac{5}{12}$ of 90
2	82 30	83 30	22 30	22 30	45 = $\frac{1}{2}$ of 90
$1\frac{1}{2}$	78 8	79 39	11 15	33 45	$67\frac{1}{2}$ = $\frac{3}{4}$ of 90
1	73 0	75 0	0 0	45 0	90 = + of 90
$1\frac{1}{2}$	66 25	68 53	11 15	33 45	$67\frac{1}{2}$ = $\frac{3}{4}$ of 90
2	57 16	60 2	22 20	22 30	45 = $\frac{1}{2}$ of 90
$2\frac{1}{2}$	53 17	56 5	26 15	18 45	$37\frac{1}{2}$ = $\frac{5}{12}$ of 90
3	48 38	51 24	30 0	15 0	30 = + of 90

In the results of the two preceding Tables, where the number of reflexions is an integer, it is easily understood how an elliptically polarized ray begins to retrace its course, and recover its state of polarization in a single plane, by the same number of reflexions by which it lost it: but it is interesting to observe, when the number of reflexions is $1\frac{1}{2}$, $2\frac{1}{2}$, $3\frac{1}{2}$, that the ray must have acquired its elliptic polarization in the middle of the second and the third reflexion; that is, when it had reached its greatest depth within the metallic surface. It then begins to resume its state of polarization in a single plane, and recovers it at the end of 3, 5, and 7 reflexions. This stationary point at which the retrograde effect commences, may be made to have its position at any depth beneath the surface, by changing the angles of some of the reflexions, or by combining plates of metal of different polarizing powers.

The same curious property is exhibited in total reflexions, as I have found that the circular polarization can be produced by $2\frac{1}{2}$, $3\frac{1}{2}$, &c. reflexions.

Hitherto we have chiefly examined the phenomena when the reflexions are performed either all above or all below the polarizing angle. We shall now proceed to the case when one reflexion is made on one side, and one on the other side of the maximum polarizing angle.

When a ray polarized + 45° has been reflected once from steel at an angle of 85° or of 54°, it has acquired partially the state of elliptic polarization, and to such a degree that three reflexions more at the same angle will complete the effect. But if the ray partially polarized elliptically by one reflexion at 85° suffers a second reflexion at 54°, it does not acquire more elliptic polarization,

but it retraces its course, and recovers its state of single polarization. The same phenomenon occurs at the following angles.

Angles of partial Elliptic Polarization.	Values of ϕ.	Angles at which it recovers its Polarization.	Values of ϕ.
1 Reflex. at $87\frac{1}{2}^\circ$	$36^\circ\ 5'$	1 Reflex. at 41°	$36^\circ\ 11'$
——— 85	27 28	——— 54	28 0
——— 80	12 12	——— 68	12 36
——— 77	5 24	——— 72	5 59
——— 75	0 0	——— 75	0 0

It is obvious, by comparing these angles with those in the preceding Table, that they correspond, and are those at which equal phases or rotations are produced.

The effect of two reflexions, at angles of equal phase, upon the inclination I of the plane of polarization is shown in the following Table.

		Inclination I of the Plane of Polarization. Observed.	Calculated.
1 Reflex. at 90°	and 1 at 0°	45°	$45^\circ\ 0'$
——— $87\frac{1}{2}$	——— 41	—	30 0
——— 85	——— 54	26	26 5
——— 80	——— 68	19	20 8
——— 77	——— 72	18	17 2
——— 75	——— 75	17	17 0

The last column of the table is calculated by the formula

$$I = \tan \varphi\,(45^\circ - i') + i',$$

i being 17°, or the inclination after two reflexions at the maximum polarizing angle.

In the preceding inquiry we have considered only the phenomena when the consecutive reflexions are performed in coincident planes. The investigation becomes more troublesome, and the results more interesting when the plane of the second reflexion is presented in every different azimuth to the ray that is either wholly or partially elliptically polarized by the first reflexion.

Let a pencil be elliptically polarized by one reflexion from steel at 75°, and let the azimuths be reckoned from the plane of this reflexion. We have already seen that a second reflexion at 75° in azim. 0° and 180° restores the pencil to a single plane of polarization; but if we turn the plane of the second

reflexion into azim. 45° or 225°, we shall find that the angle of restoration is no longer 75°, but 78°. At azim. 90° and 270° it is again 75°, and in azim. 135° and 315° it is only 68°, having varied from 68° to 78°.

The following Table shows the observed and calculated angles of restoration in different azimuths.

Azimuths from Plane of first Reflexion.	Angles of Restoration from Steel.	Complement of Angles of Restoration, or Elliptical Radii. Observed.	Calculated.
° 0 and 180	° 75	° 15	° 14.9
22½ . 202½	77	13	12.7
45 . 225	78	12	12
67½ . 247½	77¾	12¼	12.7
90 . 270	75	15	14.9
112½ . 292½	70	20	19
135 . 315	68	22	22
157½ . 337½	70	20	19
180 . 360	75	15	14.9

The radii in the two last columns are obviously those of a curve approaching to an ellipse whose major and minor axes are situated, the one 45° to the right, and the other 45° to the left of the plane of the first reflexion. The major semiaxis is 22°, and the minor 12°. Hence calling x the variable radius of the ellipse, a the greater and b the lesser semiaxis, and θ the azimuth, reckoned from the lesser axis, in which the radius x is wanted, we shall have

$$x = \frac{a\,b}{\sqrt{a^2\cos^2\theta + b^2\sin^2\theta}}.$$

When $\theta = 45°, 135°$, &c. $\sin^2\theta\cos^2\theta = \frac{1}{4}$ and $x = \frac{a\,b}{\sqrt{\frac{1}{2}a^2 + \frac{1}{2}b^2}}$.

By calculating the values of x corresponding to the azimuths in the Table, we obtain the numbers in the last column, which are so near the observed numbers as to leave no doubt that an ellipse represents the observations.

If we perform the same experiments with a plate of silver at 73°, we shall observe, with surprise, that the angle of restoration is the same in all azimuths, that is, that the ellipse has merged into the circle. There is a slight deviation indeed, just sufficient to show that the circle is slightly oval, but I could not measure the amount of it.

This result arises from the elliptical polarization of silver being very nearly circular. If we call β the angle of restoration after two reflexions,

the ratio of a to b, the major and minor axes of the ellipse, may be thus expressed:

$$a : b = \sin 2\beta : \text{rad.}$$

In steel, where $\beta = 17°$ and $2\beta = 34°$, we have $a : b = 0.559 : 1 = 12 : 21\frac{1}{2}$, differing very little from 12 : 22 the actual ratio.

In silver, where $\beta = 39° 48'$, $a : b = 0.9835 : 1 = 17 : 17\frac{1}{4}$.

In circular polarization, where $\beta = 45°$, $a : b = 1 : 1$, which gives a circle.

In rectilineal polarization, where $\beta = 0$, $a : b = 0 : 1$, which gives a straight line.

It now becomes an interesting subject of inquiry to ascertain the form and position of the ellipse, when the angle of incidence on the first plate exceeds or falls below the maximum polarizing angle.

The following experiments were made with silver at angles of incidence of 80° and 68°, the maximum polarizing angle being 73°.

SILVER.—Angle of Incidence on First Plate 80°.

Azimuth to Right.	Complement of Angle of Restoration by Second Plate.	Azimuth to Left.	Complement of Angle of Restoration by Second Plate.
°	° ′	°	° ′
0	28 2	0	28 2
$11\frac{1}{4}$	26 35	$11\frac{1}{4}$	24 40
$22\frac{1}{2}$	25 20	$22\frac{1}{2}$	21 0
$33\frac{3}{4}$	21 13	$33\frac{3}{4}$	16 40
45	18 20	45	14 35
$56\frac{1}{4}$	14 20	$56\frac{1}{4}$	11 10
$67\frac{1}{2}$	11 32	$67\frac{1}{2}$	10 0
$78\frac{3}{4}$	10 15	$78\frac{3}{4}$	10 0
90	10 0	90	10 0

SILVER.—Angle of Incidence on First Plate 68°.

0	13	0	13
$11\frac{1}{4}$	14	$11\frac{1}{4}$	13
$22\frac{1}{2}$	$15\frac{1}{2}$	$22\frac{1}{2}$	$13\frac{1}{2}$
$33\frac{3}{4}$	16	$33\frac{3}{4}$	14
45	17	45	$14\frac{1}{2}$
$56\frac{1}{4}$	19	$56\frac{1}{4}$	$15\frac{1}{2}$
$67\frac{1}{2}$	20	$67\frac{1}{2}$	$16\frac{1}{2}$
$78\frac{3}{4}$	20	$78\frac{3}{4}$	18
90	20	90	20

In the first of these sets of experiments, the semiaxes of the ellipse are as 10°

to 28°, and its major axis is in azim. 0° and 180° or in the plane of the first reflexion.

In the second series the ratio of the semiaxes is as 13° to 20°, and the major axis is in azim. 90° and 270°, or perpendicular to the plane of the first reflexion; but in both series there is a want of symmetry in the curve to the right of azim. 0° where it bulges out, showing that in both series the greater axis is a little to the right of azim. 0°.

Hence it appears that in silver, whose elliptic polarization is nearly circular, the ellipse which regulates the angles of restoration has its greater axis in the plane of the first reflexion for all angles greater than 73°, the maximum polarizing angle; and from a circle it increases in ellipticity till at the limit of 90° the lesser semiaxis is 0°, and the greater 90°, and it becomes a straight line. For angles above 73° the ellipse has its greater axis perpendicular to the plane of reflexion, and gradually increases in ellipticity from the circle till at the limit of 0° its lesser semiaxis is 0°, and its greater 90°, when it becomes a straight line.

The peculiar character of elliptic polarization shows itself in another manner, and with peculiar interest, in the variable position of the ellipses which regulate the angles of restoration upon steel.

We have already seen that the curve which is circular in silver at the maximum polarizing angle, is in steel an ellipse whose semiaxes are as 12° to 22°, the greater axes being inclined 45° to the right of azim. 0°.

The following Table will show how the effect varies at angles of incidences above and below the polarizing angle.

STEEL.—Angle of Incidence 80°.

Azimuth to Right.	Complement of Angle of Restoration by Second Plate.	Azimuth to Left.	Complement of Angle of Restoration by Second Plate.
°	°	°	°
0	23	0	23
11½	25	11½	20
22½	26	22½	16½
33¾	24	33¾	13
45	20½	45	11½
56¼	18	56¼	10
67½	15½	67½	9½
78¾	11	78¾	9¾
90	10	90	10

STEEL.—Angle of Incidence 68°.

Azimuth to Right.	Complement of Angle of Restoration by Second Plate.	Azimuth to Left.	Complement of Angle of Restoration by Second Plate.
0°	11°	0°	11°
11¼	24	11¼	10
22½	24½	22½	9
33¾	25½	33¾	9¾
45	26½	45	11½
56¼	25½	56¼	15
67½	20	67½	18
78¾	21	78¾	20
90	22	90	22

By comparing these results with those obtained from steel at 75°, and with the observations already made on the passage of the ellipse into a straight line, the following results may be deduced.

Angle of Incidence on first Steel Plate.	Ratio of Semiaxes of the Ellipse.	Character of the Ellipse.	Position of the greater Axis of the Ellipse.
0°	0° : 90°	Straight line.	Azim. 90° and 270°
68	9 : 26	Ellipse	—— betw. 45° and 56° to R.
75	12 : 22	Ellipse	—— 45° to R.
80	9½ : 26	Ellipse	—— 22½° to R.
90	0 : 90	Straight line.	—— 0

Hence it is obvious that the major axis of the ellipse is $45° \mp \varphi$ R to the right of 0° of azimuth, φ being computed from the formula

$$\tan \varphi = \frac{\cos (i + i')}{\cos (i - i')}.$$

There is a deviation at the incidence of 68° and 80° of some amount, but still it is scarcely without the limits of the errors of observations when common light is used. In strong lights the coincidence will doubtless be more perfect.

The best method of determining the position of the major axis, is to place the second plate at such an angle to the ray received from the first, that it may exceed by two or three degrees the angle of restoration in azim. 0°. Hence if we turn the second plate round the ray into all azimuths from 0° to 90° in the right hand quadrant where the greater axis lies, it must come into two azimuths where the restoration takes place at the same incidence. The comple-

ments of these two angles of incidence will be equal radii of the ellipse, and consequently the azimuth which bisects the two azimuths in question, will be that of the major axis of the ellipse. By increasing the angle of incidence on the second plate, other two azimuths containing equal radii of the ellipse will in like manner be found; and we might, if necessary, at last obtain an angle of incidence where the two radii coincided with the greater axis.

The position of the ellipse being thus given, we may determine it for all angles of incidence. Calling x the angle of incidence on the first plate, then we shall have four points in the ellipse as follows. The radii in azim. 90° and 270° are always $90° - x$, and the radius in azim. 0° and 180° is the complement of the angle of incidence at which φ in the last equation has the same value as at the angle x. Hence the form of the ellipse is also given.

In these experiments the polarization of the primitive ray has always been +45°. When this plane varies its position, that of the restored ray also changes, as we have already shown; but it remains to be seen what change takes place in the angles of restoration. At all angles of incidence, a variation in the plane of primitive polarization does not alter the angles of restoration or the corresponding radii of the ellipse in azim. 0°, 90°, 180°, and 270°; but at all intermediate azimuths of the second plate the angles of restoration diminish while the primitive plane varies from 45° to 0°, and increase when it varies from 45° to 90°. The following experiments show the progress of the change when the azimuths of the second reflexion are +45° and −45°.

STEEL.

Inclination of Plane of prim. Polarization.	Azimuth of Second Reflexion +45° to Right. Observed.	Calculated.	Azimuth of Second Reflexion −45° to Left. Observed.	Calculated.
°	°	° ′	°	° ′
0	0	0 0	1	0 0
5	2	2 4	2	1 0
10	4	3 46	3½	2 3
15	6	5 43	4½	3 7
20	8½	7 50	6	4 14
25	11	9 54	6	5 26
30	13	12 11	7	6 43
35	15	14 40	8	8 6
40	18	17 25	9½	9 41
45	20½	20 30	11½	11 30

These observations are represented by the formula, $\tan \theta = \tan a \,.\, \tan x$; a being the angle of restoration when θ, the inclination of the plane of primitive polarization, is 45°.

I have not given the values of θ from 45° to 90°, because it is difficult to ascertain even in strong lights when the evanescence commences. At 90° the action of the first plate is 0, so that at this limit the angle of restoration is the angle at which the elliptic polarization is no longer visible, from the smallness of the angle of incidence, an angle which varies with the intensity of the light employed.

Hitherto we have attended only to the phenomena produced by two similar metals. When the metals are dissimilar, the one silver and the other steel, I found that at the mean maximum polarizing angle of 74°, the inclination of the plane of the restored ray was 28° 30′. But $28° 24' = \frac{39° 48' + 17°}{2}$, so that the inclination is an arithmetical mean between that of silver and that of steel. By four reflexions at 74° the inclination was reduced to 14°, while by four reflexions at about 83° and 58° the inclination was 21½°, nearly equal to $\frac{28° 30' + 14°}{2}$, according to the formula in page 297. By thus combining dissimilar metals we may produce elliptic polarization of all degress of intensity intermediate between those produced by similar metals.

As the circular polarization of total reflexion is the limiting case of elliptical polarization, it becomes important to establish by experiment their intimate connexion and almost perfect similarity. Upon combining metallic and total reflexions this was at once evident; and I found in general that circular polarization of any intensity, as produced by either one or more reflexions from glass, may always be restored to rectilineal polarization by one or more metallic reflexions, provided the latter are all made at angles less than the maximum polarizing angle, and that the two classes of reflexions are performed in coincident planes.

As this takes place throughout the whole range of total reflexion from 41° to 90°, it follows that total differs from metallic reflexion in its not having two opposite kinds of circular polarization, like the two opposite kinds of elliptical polarization which take place on each side of the maximum polarizing angle of metals. But notwithstanding this, the circular like the elliptic

polarization has a maximum at about 50°, declining rapidly to zero at 41°, and on the other side slowly to zero at 90° of incidence.

When one reflexion from steel was combined with two total reflexions from glass at 54½°, the inclination of the plane of the restored ray was 30½°, an arithmetical mean between 45° that of total reflexion, and 17° that of steel, for $\frac{45° + 17°}{2} = 31°$. With silver the inclination was 42½°, and $\frac{45° + 39° 48'}{2}$ $= 42° 24'$.

If we make the metallic reflector receive the circularly polarized ray in every azimuth, we shall find that in azimuth 90° the circular polarization is compensated by a metallic reflexion above 80°. As the azimuth diminishes to 0°, this angle of compensation diminishes also, passes through 75° in the case of steel, and diminishes to a number depending on the angle of incidence at which the total reflexion is made. We are thus enabled to study the phenomena of circular polarization by the aid of metals, and to obtain results at which it would be exceedingly difficult, if not impossible, to arrive by any other method. This subject, however, presents too wide a field to be treated thus incidentally.

Sect. III.—*On the complementary colours produced by successive reflexions from the polished surfaces of metals.*

I have already given a general account of the phenomena of colour produced by successive reflexions; and I have shown that the tints thus produced are by no means the same as those of crystallized plates, as they do not rise in the scale by successive reflexions.

In my early experiments on total and metallic reflexions, I regarded the two classes of phenomena as exactly the same, mutatis mutandis; and in communicating these results to Dr. Young, I pointed out their coincidence with his theoretical views. Dr. Young noticed these experiments in the following manner*.

"Dr. Brewster has also shown that the total reflexion of light within a denser medium, and the brilliant reflexion at the surfaces of some of the metals, are capable of exhibiting some of the appearances of colour as if the

* Art. Chromatics, Supp. Encyc. Brit. p. 137.

light concerned were divided into two portions, the one partially reflected in the first instance, the other beginning to be refracted, and caused to return by the continued operation of the same power. The original interval appears to be extremely minute, but is capable of being increased by a repetition of similar reflexions as well as obliquity of incidence."

In a letter which I received from this eminent philosopher, dated March 25th 1816, he thus modifies an objection which he had previously made to my opinion, that the phenomena were owing to the interference of the light which had entered the surface with that which had suffered partial reflexion.

"The light which you suppose to have entered a little way into a reflecting surface, in the case of total reflexion, is singularly circumstanced with regard to the objection I mentioned in my last letter. I did not like the idea of supposing a surface of any kind to contain a finite space: but, in fact, if your theory should be confirmed, this objection might be greatly diminished by the consideration, that the thickness of the surface would still be like an infinitesimal of a different order from the interval corresponding to its apparent effect, being the versed sine of a curve of which that small interval is the arc, and possibly in a circle of curvature not very minute."

In continuing my experiments on this subject, I found that the colours of total reflexion did not rise in the scale by successive reflexions; and as they modified the tints of crystallized bodies by adding to, or subtracting from, them a given portion of a tint, I announced in the end of 1816, in the Journal of the Royal Institution, that I had discovered "a new species of moveable polarization, in which the complementary tints never rise above the white (the blueish white) of the first order, by the successive application of the polarizing influence*." I determined, experimentally, the angles at which this tint was successively produced and destroyed, and thus discovered some of the leading properties of total reflexion, before, I believe, M. Fresnel had made any experiments on the subject. It was he, however, who ascertained that this new species of polarization was circular polarization; and it is impossible to speak too highly of the ingenuity and talent which he exhibited in that difficult inquiry.

This view of the phenomena of total reflexion unsettled the opinions which I

* Journ. Roy. Inst. vol. iii. p. 213.

had entertained respecting the action of metals, and I was thus led to revise and extend the unpublished experiments which I had made on the subject.

In order to ascertain the effect of a single metallic surface, I took a crystallized plate of glass whose central tint was the blueish white of the first order, and positive like sulphate of lime. This tint varied from a quarter of a tint in value down to zero. The primitive ray was polarized $+45°$, and the plate of steel was horizontal. This ray was received at an incidence near 90°, and the principal section of the analysing prism was in the plane $+45°$, while the length of the plate of glass was fixed perpendicular to the plane $+45°$, or to the principal section of the prism, so as to move along with it.

At an incidence of 88° the metallic action destroyed the action of the equivalent crystallized plate when the section of the analysing prism was turned from $+45°$ to $+38°$.

At an incidence of $83\frac{1}{2}°$ the same effect was produced when the same section was turned into the plane $+22\frac{1}{2}°$.

And at an angle of 75°, viz. the maximum polarizing angle, the compensation took place when the axis of the crystal had moved round 45°.

In like manner, at an angle of 60° the compensation took place when the axis of the crystal was turned round $45° + 22\frac{1}{2}°$, or $-22\frac{1}{2}°$; and,

At an angle of incidence of 40° the compensation was effected when the axis of the crystal had turned round $45° + 37°$, or into the plane $-37°$. The same results are obtained when the light falls on the metal before it passes through the crystal.

Hence it follows, that at the maximum polarizing angle the effect of the equivalent crystal placed in azimuth 45° to the plane of primitive polarization, is compensated by the action of the metallic surface, while at greater angles of incidence the compensation is effected in azimuths less than 45°; and, at less angles of incidence, in azimuths greater than 45°.

When the reflexion from the metal is made in a plane perpendicular to the meridian, the opposite effect is produced.

The angles at which the compensation takes place in the preceding experiments are obviously such, that calling R the angle of rotation of the axis of the crystal, it has always to i the angle of incidence the same relation as in the formula, $\tan(45° - R) = \frac{\cos(i + i')}{\cos(i - i')}$.

Hence we are led to the important conclusion, that the pencil which enters the metal follows the changes of polarization of the partially reflected pencil, which is regulated by the same law as in transparent bodies.

It now became interesting to examine the effect produced by the joint action of the metal, and an equivalent crystal, in changing the plane of polarization of the restored ray. The following are the results with different metals at the maximum polarizing angle.

Metals.	Position of the Plane of Polarization.	Rotation effected.
	°	°
Silver (pure)	+ 42	3
Copper	+ 36½	8½
Mercury	+ 35	10
Platina	+ 34	11
Speculum metal	+ 32	13
Steel	+ 30½	14½
Lead	+ 26	19
Galæna	+ 17½	27½

These metals follow the same order in their action upon the plane of polarization that they hold in the Table in page 294, though in reference to the rotation actually produced in both cases the order is inverted.

The preceding Table points out in a very instructive manner the difference between the action of a metallic surface and an equivalent crystallized film. When two metallic surfaces act together, the plane of polarization of the restored ray is invariably thrown beyond the plane of reflexion; whereas in the combination of a crystallized film with a metallic surface, the same plane never reaches the plane of reflexion, the plane having always a negative position in the former case, and a positive one in the latter. Thus in two reflexions from silver at 73°, the primitive ray polarized + 45° has its plane of polarization changed into − 39° 48′, whereas in the combination of one reflexion from silver with the crystallized film, the plane is changed only into +42°.

In order to determine the law of the metallic action at different incidences and with different numbers of reflexions, I interposed between the eye and the metal, which was silver, a plate of calcareous spar, which exhibited its uniaxal system of rings.

Fig. 2.

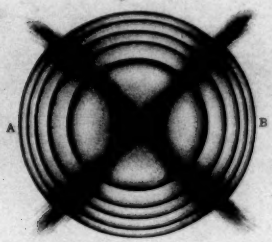


Fig. 1.

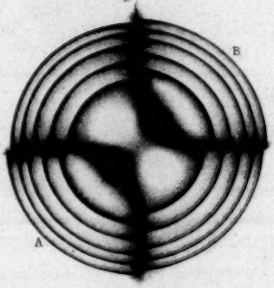
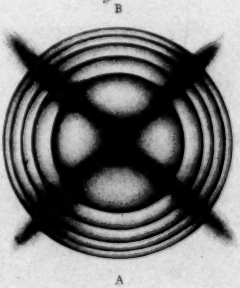


Fig. 3.



The influence of the metal in modifying the rings was a maximum at 73°, as shown in Plate XI. fig. 1. exactly as if they had been crossed by a positive crystalline film which polarized a quarter of a tint, or the pale blueish white of the first order, and whose axis was situated in a plane +45°, or that which bisects the planes of the two pencils oppositely polarized by the metal. The influence of the metal, or the tint which it polarizes, diminishes gradually from 73° to 90°, where it vanishes, and consequently where the rings recover their symmetry and their tints. At this limit the position of the axis of the equivalent film is A B (fig. 2.), a line still bisecting the planes of the two oppositely polarized pencils. In fig. 2. the rings are not represented of their own shape, but just as they are beginning to be invaded by the metallic action as at an incidence of 86° or 87°. At incidences from 73° to 0° the opposite effect takes place, the rings recovering their symmetry at 0°, and the position of the axis of the equivalent film being now vertical, and bisecting the planes of the two oppositely polarized pencils. The form of the rings before they recover their symmetry is shown in fig. 3.

At all intermediate angles of incidence the axis A B has intermediate positions; and calling A the inclination of the axis to the plane of reflexion, we shall have $A = \varphi + 45°$,

φ being positive or + from 90° to 73°, and negative or − from 73° to 0°.

The intensity of the metallic tint, so to speak, or of the positive equivalent plate T, will be $T = \frac{1}{4}\frac{P}{90} = \frac{2R}{360} = \left(\frac{45° - \varphi}{180}\right)$.

Hence we see the error of the proposition hitherto maintained, that an increase of incidence, reckoning from the perpendicular, produces the same effect as an increase of thickness in thin crystallized plates.

When the rings are combined with two reflexions at 73° in silver, or 75° in steel, they do not suffer the slightest change, the principal section of the prism being placed in the plane −39° 48′ with silver, and −17° with steel. By two reflexions, however, between 73° or 75° and 90°, an effect is produced on the rings which increases gradually in silver from 73° to 82° 30′, and diminishes from 82° 30′ to 90°. At 82° 30′ the effect is the same as after a single reflexion at 73°; for since four reflexions at 82° 30′ restore the elliptically polarized ray, two reflexions at the same angle must have produced complete elliptical polari-

zation. At angles between 82° 30′ and 90° the pencil is only partially polarized elliptically; whereas from 82° 30′ to 73° the light has been more than elliptically polarized, the restoration of it having been begun during the second reflexion. Hence, in order to determine the phase for any angle between 82° 30′ and 90°, we must take the sum of the phases for each reflexion, or 2 P; whereas between 82° 30′ and 73° we must take the excess of the sum of the two phases above 90° or 90° − 2 P. In both cases the pencil has suffered a partial elliptic polarization;—in the former, from the sum of the actions of the two reflexions, and in the latter, from their unbalanced actions. The very same effects take place between 73° and 57° 16′, the other maximum, as between 73° and 82½°; and between 57° 16′ and 90°, as between 82½° and 90°.

In the case of three reflexions there are two points or nodes of restoration, viz. 78° 8′ and 66° 35′, the maxima being at 85° 6′, 73°, and 48° 38′, at each of which points the phase is 90°. At 73° the second reflexion restores the ray elliptically polarized by the first reflexion, and the third reflexion again produces elliptic polarization. At 85° 6′ and 48° 38′, six reflexions produce a restoration of the pencil, and consequently three reflexions must have polarized the pencil elliptically with a phase of 90°. From 85° 6′ to 90°, and from 48° 38′ to 0°, the pencil has been only partially elliptically polarized, and the phase at any angle between these will be 3 P. At any angle between 48° 38′ and 85° 6′, the phase will be $\overline{2 \times 90} - 3$ P.

In general, calling n the number of reflexions, the phase between 90° and the nearest maximum, and between 0° and the nearest maximum, will be n P, while at all other angles of incidence it will be $(\overline{n-1} \times 90) - n$ P.

In order to give a general view of the number of points of restoration, and of the other phenomena which take place after different numbers of reflexions, I have drawn up the following Tables.

TABLE I. Showing the numbers of reflexions from silver at which elliptically polarized light is restored to a single plane of polarization, with the corresponding angles of incidence, and the position of the plane of restoration in relation to the plane of reflexion, computed for 20 reflexions.

(For angles less than the maximum polarizing angle.)

No. of Reflexions.		Integer Multiples.	Angle of Restoration.	No. of Reflexions.		Integer Multiples.	Angle of Restoration.
			° ′				° ′
−2	2	+4−6+8−10+12−14+16−18+20	71 0	3½	3.8	+19	58 15
2½	2.111	+19	71 42	− 4	4.0	+8−12+16−20	57 16
2½	2.125	+17	71 32	4½	4.25	+17	55 54
2½	2.143	+15	71 23	4½	4.333	+13	55 29
2½	2.167	−13	71 8	4½	4.5	−9+18	54 42
2½	2.200	+11	70 48	4⅔	4.667	−14	53 54
2½	2.222	−20	70 34	4¾	4.75	−19	53 31
2½	2.25	−9+18	70 17	+ 5	5	+10+15+20	52 27
2⅔	2.286	−16	69 53	5½	5.333	−16	51 5
2½	2.333	+7+14	69 29	5½	5.5	−11	50 27
2½	2.375	−19	69 3	5⅔	5.667	−17	49 49
2½	2.4	−12	68 59	− 6	6	+12−18	48 38
2½	2.428	−17	68 33	6½	6.333	+19	47 23
2½	2.5	−5+10−15+20	67 54	6½	6.5	−13	46 57
2⅔	2.571	−18	67 14	6⅔	6.667	−20	46 32
2½	2.6	+13	66 58	+ 7	7	+14	45 35
2½	2.667	−8+16	66 25	7½	7.5	−15	44 13
2½	2.714	+19	66 0	− 8	8	+16	43 0
2½	2.75	−11	65 45	8½	8.5	+17	41 52
2⅔	2.8	−14	65 23	+ 9	9	+18	40 51
2½	2.833	+17	65 9	9½	9.5	−19	39 51
2½	2.857	−20	65 0	−10	10	+20	39 0
+3	3	+6+9+12+15+18	63 43	+11	11		37 15
3½	3.167	−19	62 29	−12	12		35 50
3½	3.2	−16	62 15	+13	13		34 33
3½	3.25	−13	61 55	−14	14		32 30
3½	3.33	−10	61 20	+15	15		32 15
3½	3.4	−17	60 53	−16	16		31 17
3½	3.5	−7+14	60 15	+17	17		30 30
3½	3.6	−18	59 38	−18	18		29 42
3½	3.667	+11	59 13	+19	19		28 56
3½	3.75	−15	58 42	−20	20		28 10

TABLE II. Showing the numbers of reflexions from silver at which elliptically polarized light is restored to a single plane of polarization, with the corresponding angles of restoration, and the position of the plane of restoration in relation to the plane of reflexion, computed for 20 reflexions.

(For angles greater than the maximum polarizing angle.)

No. of Reflexions.		Integer Multiples.	Angle of Restoration.
			° ′
−2	2	+4−6+8−10+12−14+16−18−20	73 0
2½	2.111	−19	74 9
2½	2.125	+17	74 18
2½	2.143	−15	74 28
2½	2.167	+13	74 44
2½	2.200	−11	75 0
2½	2.222	−20	75 12
2½	2.256	+9+18	75 26
2½	2.286	−16	75 43
2½	2.333	−7+14	76 2
2½	2.375	+19	76 19
2½	2.4	−12	76 33
2½	2.428	+17	76 44
2½	2.5	+5+10+15+20	77 13
2½	2.571	−18	77 38
2½	2.6	−13	77 48
2½	2.667	−8+16	78 38
2½	2.714	−19	78 23
2½	2.75	+11	78 33
2½	2.8	−14	78 47
2½	2.833	+17	78 57
2½	2.857	−20	79 4
−3	3.0	+6−9+12−15+18	79 40
3½	3.167	+19	80 37
3½	3.2	−16	80 24
3½	3.25	+13	80 34
3½	3.333	−10+20	80 50
3½	3.4	−17	81 2
3½	3.5	+7+14	81 19
3½	3.6	−18	81 35
3½	3.667	−11	81 45
3½	3.75	+15	81 57
	3½	3.8	−19	82 8
−4	4	+8−12+16−20	82 30
	4½	4.25	+17	82 58
	4½	4.333	−13	83 16
	4½	4.5	+9+18	83 23
	4½	4.667	−14	83 38
	4½	4.75	+19	83 45
−5	5	+10−15+20	84 5
	5½	5.333	−16	84 27
	5½	5.5	+11	84 38
	5½	5.667	−17	84 48
−6	6	+12−18	85 6
	6½	6.333	−19	85 22
	6½	6.5	+13	85 30
	6½	6.667	−20	85 36
−7	7	+14	85 49
	7½	7.5	+15	86 7
−8	8	+16	86 21
	8½	8.5	+17	86 35
−9	9	+18	86 46
	9½	9.5	+19	86 56
−10	10	+20	87 5
−11	11		87 20
−12	12		87 35
−13	13		87 46
−14	14		87 56
−15	15		88 4
−16	16		88 11
−17	17		88 18
−18	18		88 24
−19	19		88 28
−20	20		88 33

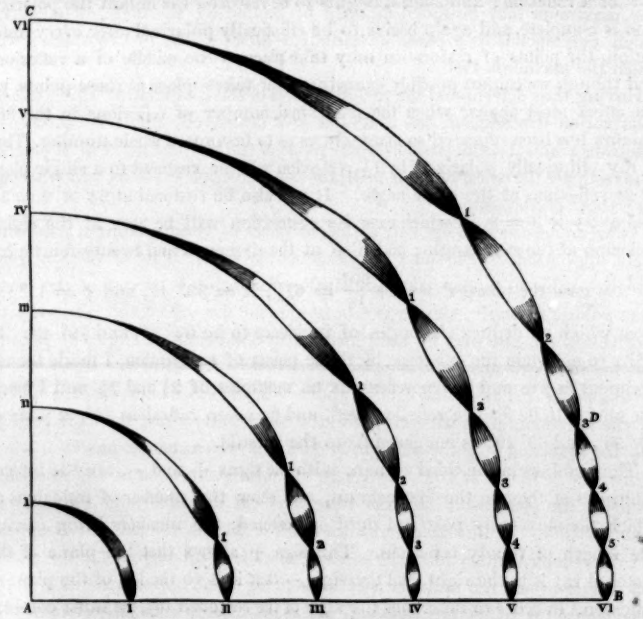
The first column of the preceding Tables shows the smallest number of reflexions at which a pencil of elliptically polarized light is restored to a single plane of polarization at the angle contained in the fourth column; and consequently the half of these numbers is the number of reflexions at which light is elliptically polarized at the same angle. Thus at three reflexions the ray is

restored to a single plane of polarization at 63° 43′, and 79° 40′, and consequently at $1\frac{1}{2}$ reflexion it is elliptically polarized at that angle. This is easily understood when the number of reflexions is an integer; but it requires some explanation when the number is partly fractional. It has been already stated, in page 301, that elliptical polarization may be completed at any fractional part of a reflexion; and since it begins to be restored the instant the polarization is complete, and again begins to be elliptically polarized after every restoration, the points of restoration may take place in the middle of a reflexion; and though we cannot possibly examine what takes place at these points, yet the effect must appear when the fractional number of reflexions in the first column has been repeated so many times as to become a whole number. Thus a ray elliptically polarized by $1\frac{1}{3}$ reflexion will be restored to a single plane at $2\frac{1}{4}$ reflexions at the same angle. It will also be restored at $2\frac{1}{4} \times 2 = 3\frac{1}{2}$, and at $2\frac{1}{4} \times 3 = 8$, in which case its restoration will be seen at the eighth reflexion at the same angle; and also at the sixteenth and twenty-fourth, &c. In this case the phase P will be $\frac{90°}{1\frac{1}{3}} = 67\frac{1}{2}°$, R = 33° 45′, and φ = 11° 15′, from which we deduce the angles of incidence to be 63° 43′, and 79° 40′. In order to ascertain the existence of these points of restoration, I made the experiment at five and seven reflexions as multiples of $2\frac{1}{2}$ and $2\frac{1}{3}$, and I found the angles to be for five reflexions 68°, and for seven reflexions 70°, in place of 67° 54′, and 69° 29′, as computed from the formula.

The numbers in the third column, with the signs + and −, are the integer multiples of those in the first column, and show the number of reflexions at which the elliptically polarized light is restored, the numbers being carried the length of twenty reflexions. The sign + shows that the plane of the restored ray is to the right, and the sign − that it is to the left of the plane of reflexion. In order to determine the sign of the restored ray, we must consider that in the same quadrant the signs necessarily alternate. Now at 73°, the maximum polarizing angle, the signs are −2, +4, −6, +8, −10, +12, &c.; and I have also found that all the integer numbers in column 1st, Table I. have their signs + or positive, as +3, +5, +7, +9, &c., and all the even numbers their signs − or negative, as −4, −6, −8, −10, &c.; whereas in Table II. all the integer numbers are negative whether odd or even, thus, −3, −4, −5, −6, &c.

By setting out therefore from these points, and attending to the alternation of the signs, it is easy to determine for any number of reflexions its proper signs, whether it is a multiple of an integer or of a mixed number.

In order to illustrate this Table, I have here projected some of its results as far as six reflexions. The concentric arches I I, II II, &c. represent the quadrant



of incidence for one, two, &c. reflexions, B being the point of 90°, and C that of 0° of incidence. The point D or the line A D is the point or line of maximum polarization, viz. 73° for silver; and the figures 1, 2, 3, 4, 5, &c. show the points or nodes, and their distances from C, the angles of restoration. The loops or double curves lying between the points 1, 2, 3, are drawn to give an idea of the intensity of the elliptic polarization, which has its minimum at 1, 2, 3, &c. and its

maximum at intermediate points. These points of maximum intensity do not bisect the loops, or are not equidistant from the minima 1, 2, &c.; but such is their relation to them, that the maximum for n reflexions is the minimum for $2n$ reflexions corresponding to the same angle. Thus the maximum for one reflexion, viz. 73°, is the minimum for two reflexions; and the maxima for two reflexions, viz. 82° 30′ and 63° 43′, are the minima for four reflexions. The maximum may be found directly by computing the angle of incidence, which corresponds to a phase intermediate between the two minima, within which the maximum lies.

Having thus determined the various points of the quadrant, at which elliptic polarization is produced, and at which it is destroyed, after any number of reflexions; and also the position of the plane of the restored ray, I shall proceed to investigate the cause of those brilliant complementary colours which accompany these phenomena.

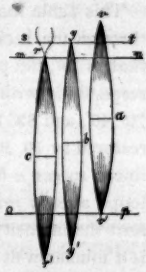
As all transparent bodies have different values of their maximum polarizing angle, appropriate to the index of refraction for each colour of the spectrum, it is reasonable to suppose that as elliptic polarization is effected at the maximum polarizing angle, this angle would vary for the differently coloured rays. That this is the case may be easily proved by observing the angles of restoration for homogeneous light after two reflexions. In silver the difference of the angles for red and blue light is about 5° in the sun's rays; so that calling 73° the maximum polarizing angle for the mean yellow ray, the angle will be $70\frac{1}{2}°$ for blue, and $75\frac{1}{2}°$ for red light. Hence if we examine a pencil of white light twice reflected at $70\frac{1}{2}°$, and place the principal section of the analysing prism in the plane $-39° 48′$, the blue rays will disappear and the red will remain visible. In like manner, at an angle of 75° 30′ the red will disappear, and the complementary blue will be visible; while at an angle of 73° the yellow will disappear and red and blue will be seen together, one on each side of the place where the yellow has vanished. At angles of incidence greater than $75\frac{1}{2}°$ and less than $70\frac{1}{2}°$, and also at intermediate angles, the blue or the red wil still predominate in the pencil, the blue being in excess at all angles greater than 73°, and the red in excess at all angles less than 73°. Such are precisely the phenomena which take place, as will appear from the following Table.

Angle of Incidence of the two Reflexions.	Colours with ordinary Light.
63° . .	Very pale yellow, growing whiter at less incidences
64 . .	Pale yellow.
65 . .	Pale saffron yellow.
66 . .	Saffron yellow.
67 . .	Paler orange yellow.
68 . .	Orange yellow.
69 . .	Reddish orange.
70 . .	Tile red.
70½ . .	Vermilion red.
71 . .	Scarlet.
72 . .	Bright pink.
73 . .	Dark pink.
74 . .	Deep China blue.
75 . .	Indigo.
75½ . .	Pure bright blue.
76 . .	Paler blue.
77 . .	Whitish blue.
78 . .	Blueish white, growing white at greater angles.

It is obvious from what has been already stated, that with homogeneous yellow light the pencil will not vanish in passing from 73°, where it is evanescent, to 90°, and 0° where it is also evanescent; but the intensity of the extraordinary pencil of the analysing rhomb will increase from 0° to half the reflected light, from 73° to 82½°, and from 73° to 57° 16′, and will decrease from the same points to 90° and 0°. The same is true of the red and blue rays, the former having its maximum intensity at an angle greater than 82½° and greater than 57° 16′, and the latter at an angle less than 82½° and less than 57° 16′.

In order to ascertain the phenomena in homogeneous light, let us suppose that polarized yellow light suffers four reflexions from silver, and let us consider what should take place in the loop 2, 3 of the quadrant IV, IV. (See Fig. p.318.) At the node 2, or 73°, the inclination of the restored pencil is +31° 52′, and at the node 3, or 82° 30′, it is −37° 22′, and the point of maximum between 2 and 3 is at 78° 8′. If at 73° we place the principal section of the analysing prism

in the plane $+31° 52'$ the extraordinary ray will vanish, and the light will pass into the ordinary image; and if at $82° 30'$ we place it in $-37° 22'$, the same effect will be produced. At 74° a small portion of light will pass into the extraordinary image, and this portion will gradually increase to $78° 8'$, the principal section of the prism having been turned round gradually from $+31° 52'$ to 0°, as described in page 291. The ordinary and extraordinary images now approach most to equality, and they vary in intensity according to the same law in passing from $78° 8'$ to $82° 30'$, the axis of the prism having now come into the plane $-37° 22'$. The very same phenomena take place with red and blue light, only the points of restoration and the maximum occur at different angles of incidence, so that the spaces between the minima have different lengths for the differently coloured rays. These spaces or loops, therefore, will overlap each other, as will be understood from the annexed diagram, where they are shown separately, $r\,r'$ being the red loop, $y\,y'$ the yellow, $v\,v'$ the violet one, the points r, y, v, r', y', v' the minima or nodes, and a, b, c the maxima. When these loops are viewed superposed as when they form white light, then the tint in the extraordinary image will be white, minus the three quantities of light that have disappeared from the extraordinary ray. At the line $m\,n$, passing through the node of the red loop, the red will have vanished, and the mixture of the yellow and the violet which remains will constitute a greenish blue pencil, decreasing in its blue tint towards a, and becoming pink, and then red towards $t\,s$, in consequence of part of the light of the other red loop above r now passing into the extraordinary ray. At v and at v', where the violet disappears, the mixture of the yellow and the red will form an orange pencil, which will be reddest at v and v', and shading off to white at a. At the line $s\,t$ the yellow vanishes, and across the upper part of the luminous disc, there will be light with an excess of red, and across the lower part of it, light with an excess of blue. This takes place with even numbers of reflexions; with odd numbers the blue light is uppermost and the red undermost.

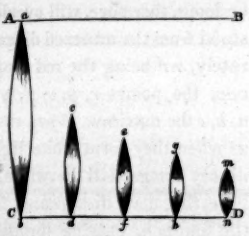


The phenomena of colour, as seen by white light, vary greatly with the number of reflexions, both with respect to the depth of the colours themselves

and the rapidity of their changes. In order to investigate the nature of these variations, let us consider what will take place at 2, 4, 6, 8, and 10 reflexions from silver in the loops above and adjacent to 73° the maximum polarizing angle. The following are the numbers which regulate the phenomena.

Fig. 6.	No. of the Reflexions.	Nature of the Reflexions.	Limits of the Loops.	Length of the Loops.	Inclination of the Plane, or ϕ.
			° ° ′	° ′	° ′
a b	2	First of the series	73 — 90 0	17 0	39 48
c d	4	First of the series	73 — 82 30	9 30	37 22
e f	6	Multiple of 3	73 — 79 40	6 40	32 25
g h	8	Multiple of $2\frac{2}{3}$	73 — 78 8	5 8	27 53
m n	10	Multiple of $2\frac{1}{2}$	73 — 77 13	4 13	24 16

This Table may be illustrated by the annexed diagram, where A B passes through the incidence of 90°, and C D through that of 73°, the points *m*, *g*, *e*, *c*, *a* corresponding respectively with the incidences of 77° 13′, 78° 8′, 79° 40′, and 82° 30′, or those at which the ray is restored by 10, 8, 6, and 4 reflexions. The curvilineal spaces *a b*, *c d*, *e f*, *g h*, and *m n*, are the loops already referred to, whose breadths represent the intensity of the extraordinary ray, which is a minimum at the nodes *a*, *c*, *e*, *g*, *m*, and *b*, *d*, *f*, *h*, *n*, and reaches its maximum near the middle of the loops.



If the image reflected from the silver is a circular disc of white light of a given magnitude, then by two reflexions at 73°, or at the point *b* the extraordinary image will be red above and blue below, when the principal section of the analysing prism is in the plane — 39° 48′; but these colours will be very faint, as the disc occupies but a small part of the loop *a b*. The disc indeed may be made so small, that the extraordinary image will entirely disappear in this loop. In this case the ordinary image will be white, as all the reflected light will pass into it. At four reflexions the loop *c d* is little more than one half of *a b*, and consequently the light will vary much more rapidly from *d* to the maximum. When the analysing prism has its principal section in the plane −37° 22′, the extraordinary image at *c* will be coloured with red light above and blue below; and when it is in the plane + 31° 52′, the extraordinary image at *d* will be

similarly coloured: The colours will be much brighter than in the case of two reflexions, and consequently the extraordinary image will not vanish. The consequence of this is, that the ordinary image is not white as before, but yellow, because a considerable portion of red and blue light are left in the extraordinary image.

As the number of reflexions increase, and the loops $e f$, $g h$, &c. diminish, the disc will occupy a greater proportion of the whole loop, and the red and blue colours with which it is crossed grow brighter and brighter, and come closer and closer to their line of junction in the middle of the disc. Hence a greater quantity of red and blue light is left out of the ordinary image, which on this account becomes yellower and yellower, and at last of a greenish hue.

In order to determine the position of the principal section of the analysing prism, when the extraordinary image is a minimum for any angle of incidence α, and any number of reflexions, let $\psi, \chi =$ the inclinations of the plane of polarization of the restored ray at the nodes a, b; $m, n =$ the inclinations or values of φ in the formula $\tan \varphi = \frac{\cos(i + i')}{\cos(i - i')}$ suited to the angles of incidence at the nodes; $x =$ the inclination φ suited to the incidence α.

Now it is obvious that at the one node, the position of the principal section of the analysing prism, when the extraordinary image is a minimum, is $+ \psi$, and that it gradually changes to 0° and then passes to $- \chi$, thus undergoing a change equal to $\psi + \chi$, while the inclination φ varies by a quantity equal to $m - n$. Hence calling I the inclination of the principal section to the plane $+ \psi$ at the angle of incidence α, we have $m - n : \psi + \chi = m - x : \mathrm{I}$.

$$\text{Hence } \mathrm{I} = \psi + \chi \left(\frac{m - x}{m - n}\right)$$

$$\text{When } x = n, \ \mathrm{I} = \psi + \chi.$$

$$\text{When } x = \frac{m - n}{2}, \quad \frac{m - x}{m - n} = \tfrac{1}{2} \quad \text{and } \mathrm{I} = \frac{\psi + \chi}{2}.$$

When the nodes of the loop are on different sides of the maximum polarizing angle, which happens only in the middle loop of 3, 5, 7, &c. reflexions, then m and n have opposite signs, and consequently their difference is $m + n$, and, as in this case $m = n$, the formula becomes $\mathrm{I} = \psi + \chi \left(\frac{m - x}{2m}\right)$.

It is impossible to determine the relative intensities of the ordinary and ex-

traordinary image at any angle α, because this must depend on the relative intensities of the pencils by whose interference the elliptical polarization is produced. In silver these pencils approach to equality, but in steel and other metals they are very unequal.

Having thus shown how to determine the phenomena of elliptic polarization for any angle of incidence, for any number of reflexions, and for homogeneous light of any colour, I shall conclude this paper with some observations on a very remarkable anomaly which has presented itself in the course of this inquiry.

The phenomena which have been described, indicate very clearly that the angle of maximum elliptic polarization for one reflexion, or the angle of restoration after two equiangular reflexions, is the maximum polarizing angle of the metal, and consequently that its tangent is the index of refraction, as shown in the following Table*.

Names of Metals.	Angles of Maximum Polarization.	Index of Refraction.
Grain tin	78° 30′	4.915
Mercury	78 27	4.893
Galæna	78 10	4.773
Iron pyrites	77 30	4.511
Grey cobalt	76 56	4.309
Speculum metal	76 0	4.011
Antimony melted	75 25	3.844
Steel	75 0	3.732
Bismuth	74 50	3.689
Pure silver	73 0	3.271
Zinc	72 30	3.172
Tin plate hammered	70 50	2.879
Jewellers' gold	70 45	2.864

* This Table completely proves that the refractive index of metals cannot be deduced from their reflective power; for silver, which surpasses them all in reflective power, stands very low in refractive power. Mr. Herschel has noticed the difference between the indices of refraction deduced by these two methods in the case of mercury, which he makes 5.829 as given by its reflective power, and 4.16 as given by its polarizing angle. He makes the index for steel 2.85. When we consider that metals reflect the light that enters their substance, it must be obvious that the quantity of light which

This conclusion is not opposed by any of the phenomena, when we consider merely the mean refrangible ray to which these numbers refer: but when we use homogeneous light, a very strange anomaly occurs. The maximum angle of elliptic polarization for red light in the case of silver is 75° 30′, and for blue light 70° 30′, giving

		Angle.
Index of refraction for red light	3.866	75° 30′
——— mean ray	3.271	73 0
——— blue light	2.824	70 30

the order of the refrangibilities being inverted.

The perfect similarity between the action of metals, and the total reflexion of the second surfaces of transparent bodies, promised to throw light upon this difficulty. I accordingly examined the formula of FRESNEL for total reflexion, where the phase P is thus expressed:

$$\text{Cos P} = \frac{2m^2(\sin i)^4 - (m^2+1)(\sin i)^2 + 1}{m^2+1(\sin i)^2 - 1}.$$

From this formula it follows that when $m = 1.51$, and $i = 54° 37'$, P will be 45° for one reflexion, and consequently for two reflexions $2\,\text{P} = 90°$. If m increases as it does for blue light, then the phase will be 45° at an angle of incidence above 54° 37′, that is, the circular polarization of the pencil will take place at a greater angle of incidence for blue than for red light, which is the reverse of what takes place in metals. Upon making the experiment, however, with total reflexion, we shall find that the blue rays are circularly polarized by two reflexions at a less angle than the red rays, thus approximating the two classes of phenomena even with respect to this singular anomaly. Hence in order to accommodate M. FRESNEL's formula to homogeneous light of different colours, let m be the index of refraction for the homogeneous ray, and d the difference between it and the mean index, then the formula for the phase P will become $\text{Cos P} = \frac{2(m \pm d)^2(\sin i)^4 - ((m \pm d)^2+1)(\sin i)^2 + 1}{((m \pm d)^2+1)(\sin i)^2 + 1}$

they reflect is a function not only of their refractive power, but of their transparency, which will be proportional to the intensity of the reflected pencil that has entered the metal. If this is the case, the transparency will be proportional to the inclination of the plane of the restored ray after two reflexions at the maximum polarizing angle, and the order of the transparencies of the different metals will be that of the Table, p. 294. See Mr. HERSCHEL's Treatise on Light, § 594, 845.

the sign + being used for the red or least refrangible rays, and — for the blue or most refrangible.

For the same reason, in calculating the phases of an elliptically polarized homogeneous ray by means of the formula $\tan \varphi = \frac{\cos(i + i')}{\cos(i - i')}$, we must determine i' from the formula $\sin i' = \frac{\sin i}{m \pm d}$, the sign + being used for the red or least refrangible, and — for the blue or most refrangible rays.

As the theoretical considerations upon which M. Fresnel is said* to have constructed his formula, did not present to him the above anomaly, it would be in vain for me to seek an explanation of it. I may just mention, however, that at the second surfaces of bodies the angle of maximum polarization, or $\tan \frac{1}{m}$ is necessarily less for the least refrangible than for the mean rays, which is the reverse of what takes place at the first surface; and since the limit of total reflexion whose sine is $\frac{1}{m}$, or since the sphere of circular polarization commences sooner for the least than for the most refrangible rays, it might be expected that the angle of maximum circular polarization should be less for these rays, as I have found to be the case.

Although we do not understand the nature of the forces by which metals reflect the two oppositely polarized pencils, yet they act exactly like the second surfaces of transparent bodies when producing total reflexion. Setting out from a perpendicular incidence, the least refrangible rays begin to suffer the double reflexion sooner than the mean ray, and they sooner reach their maximum of elliptic polarization, thus exhibiting the inversion as it were of the spectrum, which we have noticed.

The theory of elliptic vibrations as given by Fresnel, will no doubt embrace the phenomena of elliptic polarization; and when the nature of metallic action shall be more thoroughly examined, we may expect to be able to trace the phenomenon under consideration to its true cause.

Allerly, *February* 19*th*, 1830.

* I am acquainted with M. Fresnel's formula only from the account given of it by Mr. Herschel.

XXII. *Researches in physical astronomy.* By JOHN WILLIAM LUBBOCK, *Esq. Fellow of the Royal Society.*

Read April 29, 1830.

IN the first volume of the Mécanique Céleste, LAPLACE has given expressions for the variations of the elliptic constants, which are true when the square and higher powers of the disturbing force are neglected; and he has proved, upon the supposition that the planets move in the same direction, in orbits nearly circular and little inclined one to another, that the eccentricities and inclinations vary within small limits, thereby demonstrating within these conditions the stability of the planetary system. But these conditions are not necessary to the stability of a system of bodies, subject to the law of attraction, which obtains in our system. I have given in the following investigation the expressions for the variations of the elliptic constants, which are rigorously true whatever power of the disturbing force be retained; and it is easy to conclude from the form of their expressions, that however far the approximation be carried, the eccentricity, the major axis, and the tangent of the inclination of the orbit to a fixed plane, contain no term which varies with the time; their variations are all periodic, and they oscillate therefore within certain limits. This theorem is no longer true if the planet moves in a resisting medium.

I have also given some equations which obtain when an angle is taken for the independent variable, which in the elliptic movement is the eccentric anomaly, which are of remarkable simplicity, and which, as far as I know, have never been noticed, and the development of the disturbing function R to the quantities involving the squares and products of the eccentricities inclusive.

Let x, y, z denote the rectangular co-ordinates of the planet m.

- r ..... distance from the sun
- r' ..... distance from the sun projected upon the plane x, y
- $^*\lambda$ ..... longitude reckoned upon the plane of its orbit
- λ' ..... longitude reckoned upon the plane $x\,y$
- s ..... tangent of the latitude
- v .... a variable, which in the elliptic theory is the eccentric anomaly
- m .... the mass
- a ..... semiaxis major
- e ..... eccentricity
- ϖ ..... longitude of the perihelion
- ε ..... longitude of the epoch
- ν ..... longitude of the ascending node
- ι ..... inclination of the orbit to the plane $x\,y$
- α .... a constant quantity which accompanies v
- M .... the mass of the sun.

$$M + m = \mu, \quad \sqrt{\frac{\mu}{a^3}} = n.$$

$x = r'\cos\lambda'$, $y = r'\sin\lambda'$, $z = r's$, and in the elliptic motion $r'^2 \mathrm{d}\lambda' = h\,\mathrm{d}t$.

$$R = m_i \left\{ \frac{x x_i + y y_i + z z_i}{\{x_i^2 + y_i^2 + z_i^2\}^{\frac{1}{2}}} - \frac{1}{\{(x - x_i)^2 + (y - y_i)^2 + (z - z_i)^2\}^{\frac{1}{2}}} \right\}$$

$$= m_i \left\{ \frac{r'\{\cos(\lambda' - \lambda'_i) + s s_i\}}{r_i'^2 (1 + s_i^2)^{\frac{1}{2}}} - \frac{1}{\{r'^2(1+s^2) - 2r'r'_i\{\cos(\lambda' - \lambda'_i) + s s_i\} + r_i'^2(1+s_i^2)\}^{\frac{1}{2}}} \right\}$$

* Laplace uses the letter v to denote longitude, u the eccentric anomaly, and φ the inclination of the orbit to a fixed plane; but as v is very frequently used to signify velocity, and φ geographical latitude, and as the letters of the Greek alphabet are generally used for angles, I have taken the letters λ, v, and ι for these quantities.

$$= -m_i \left\{ \begin{array}{l} \dfrac{1}{r_i' \sqrt{1+s_i^2}} + \dfrac{r'^2}{4 r_i'^3} \{1 + 3\cos(2\lambda' - 2\lambda_i') + 12 s s_i \cos(\lambda' - \lambda_i') - 2 s^2\} \\ \qquad + \dfrac{r'^3}{8 r_i'^4} \{3(1 - 4 s^2)\cos(\lambda' - \lambda_i') + 5\cos(3\lambda' - 3\lambda_i')\} \\ \qquad + \dfrac{r'^4}{64 r_i'^5} \{9 + 20\cos(2\lambda' - 2\lambda_i') + 35\cos(4\lambda' - 4\lambda_i')\} \end{array} \right\}$$

$$\left(\frac{dR}{dr'}\right) = m_i \left\{ \frac{\cos(\lambda' - \lambda_i') + s s_i}{r_i'^2 (1+s_i^2)^{\frac{1}{2}}} + \frac{r' (1+s^2) - r_i' \{\cos(\lambda' - \lambda_i') + s s_i\}}{\{r'^2 (1+s^2) - 2 r' r_i' \{\cos(\lambda' - \lambda_i') + s s_i\} + r_i'^2 (1+s_i^2)\}^{\frac{3}{2}}} \right\}$$

$$\left(\frac{dR}{d\lambda}\right) = -m_i \left\{ \frac{r' \sin(\lambda' - \lambda_i')}{r_i'^2 (1+s^2)^{\frac{1}{2}}} - \frac{r' r_i' \sin(\lambda' - \lambda_i')}{\{r'^2 (1+s^2) - 2 r' r_i' \{\cos(\lambda' - \lambda_i') + s s_i\} + r_i'^2 (1+s_i^2)\}^{\frac{3}{2}}} \right\}$$

$$\left(\frac{dR}{ds}\right) = m_i \left\{ \frac{r' s_i}{r_i'^2 (1+s_i^2)^{\frac{1}{2}}} + \frac{r'^2 s - r' r_i' s_i}{\{r'^2 (1+s^2) - 2 r' r_i' \{\cos(\lambda' - \lambda_i') + s s_i\} + r_i'^2 (1+s_i^2)\}^{\frac{3}{2}}} \right\}$$

Let P be the place of the planet m, S the place of the sun, S N the intersection of the orbits of m and m_i, S L the line from which longitudes are reckoned, P_i', the projection of P_i upon the plane of the orbit of P; then if the plane $x\,y$ coincide with the orbit of m, $SP = r$, $PSL = \lambda$, $NSL = \nu$, $P_i SN + NSL = \lambda_i$, $P_i' SL = \lambda_i'$, $SP_i' = r_i'$, $SP_i = r_i = r_i' (1+s_i^2)^{\frac{1}{2}}$

$$R = m_i \left\{ \frac{SP \times SP_i \cos PSP_i}{SP_i^3} - \frac{1}{\{SP^2 - 2 SP \times SP_i \cos PSP_i + SP_i^2\}^{\frac{1}{2}}} \right\}$$

$$\left(\frac{dR}{dr}\right) = m_i \left\{ \frac{SP_i' \cos PSP_i'}{SP_i^3} + \frac{SP - SP_i' \cos PSP_i'}{\{SP^2 - 2 SP \times SP_i' \cos PSP_i' + SP_i'^2\}^{\frac{3}{2}}} \right\}$$

$$r\left(\frac{dR}{dr}\right) = m_i \left\{ \frac{SP \times SP_i \cos PSP_i}{SP_i^3} + \frac{SP^2 - SP \times SP_i \cos PSP_i}{\{SP^2 - 2 SP \times SP_i' \cos PSP_i' + SP_i'^2\}^{\frac{3}{2}}} \right\}$$

$$\left(\frac{dR}{d\lambda}\right) = -m_i \left\{ \frac{SP \times SP_i' \sin PSP_i'}{SP_i^3} - \frac{SP \times SP_i' \sin PSP_i'}{\{SP^2 - 2 SP \times SP_i' \cos PSP_i' + SP_i'^2\}^{\frac{3}{2}}} \right\}$$

$$\left(\frac{dR}{ds}\right) = m_i \left\{ \frac{SP \times P_i P_i'}{SP_i^3} - \frac{SP \times P_i P_i'}{\{SP^2 - 2 SP \times SP_i' \cos PSP_i' + SP_i'^2\}^{\frac{3}{2}}} \right\}$$

$$\cos(\lambda - \lambda_i) = \cos(\lambda - \nu)\cos(\lambda_i - \nu) + \cos\iota \sin(\lambda - \nu)\sin(\lambda_i - \nu)$$

$$= \cos^2\tfrac{\iota}{2}\cos(\lambda - \lambda_i) + \sin^2\tfrac{\iota}{2}\cos(\lambda + \lambda_i - 2\nu)$$

$$\tan(\lambda_i' - \nu) = \cos\iota \tan(\lambda_i - \nu)$$

$$\sin(\lambda_i' - \nu) = \frac{\cos\iota \tan(\lambda_i - \nu)}{(1 + \cos^2\iota \tan^2(\lambda_i - \nu))^{\frac{1}{2}}}$$

$$\cos(\lambda_i' - \nu) = \frac{1}{(1+\cos^2 i \tan^2(\lambda_i - \nu))^{\frac{1}{2}}}$$

$$r r_i' \sin(\lambda - \lambda_i') = r r_i' \{\sin\lambda \cos\lambda_i' - \cos\lambda \sin\lambda_i\}$$

$$\cos\lambda_i' = \cos(\lambda_i' - \nu)\cos\nu - \sin(\lambda_i' - \nu)\sin\nu$$

$$= \frac{\cos(\lambda_i - \nu)\cos\nu - \sin(\lambda_i - \nu)\sin\nu\left(1 - 2\sin^2\frac{i}{2}\right)}{(1 - \sin^2 i \sin^2(\lambda_i - \nu))^{\frac{1}{2}}}$$

$$= \frac{\cos\lambda_i + 2\sin^2\frac{i}{2}\sin(\lambda_i - \nu)\sin\nu}{(1 - \sin^2 i \sin^2(\lambda_i - \nu))^{\frac{1}{2}}}$$

$$\sin\lambda_i' = \sin(\lambda_i' - \nu)\cos\nu + \cos(\lambda_i' - \nu)\sin\nu$$

$$= \frac{\sin(\lambda_i - \nu)\cos\nu\left(1 - 2\sin^2\frac{i}{2}\right) + \cos(\lambda_i - \nu)\sin\nu}{(1 - \sin^2 i \sin^2(\lambda_i - \nu))^{\frac{1}{2}}}$$

$\sin P_i S P_i' = \sin i \sin(\lambda_i - \nu)$, $r_i' = r_i \cos P_i S P_i' = r_i\left(1 - \sin^2 i \sin^2(\lambda_i - \nu)\right)^{\frac{1}{2}}$ therefore,

$$r r_i' \sin(\lambda - \lambda_i') = r r_i \left\{\sin(\lambda - \lambda_i) + 2\sin^2\frac{i}{2}\sin(\lambda_i - \nu)\cos(\lambda - \nu)\right\}$$

$$= r r_i \left\{\cos^2\frac{i}{2}\sin(\lambda - \lambda_i) + \sin^2\frac{i}{2}\sin(\lambda + \lambda_i - 2\nu)\right\}$$

similarly it may be shown that

$$r r_i' \cos(\lambda - \lambda_i') = r r_i \left\{\cos^2\frac{i}{2}\cos(\lambda - \lambda_i) + \sin^2\frac{i}{2}\cos(\lambda + \lambda_i - 2\nu)\right\}$$

$$S P_i = \frac{a_i(1 - e_i^2)}{1 + e_i\cos(P_i S N + N S L - \varpi_i)} = \frac{a_i(1 - e_i^2)}{1 + e_i\cos(\lambda_i - \varpi_i)}$$

$$\frac{d^2x}{dt^2} + \frac{\mu x}{r^3} + \left(\frac{dR}{dx}\right) = 0, \quad \frac{d^2y}{dt^2} + \frac{\mu y}{r^3} + \left(\frac{dR}{dy}\right) = 0, \quad \frac{d^2z}{dt^2} + \frac{\mu z}{r^3} + \left(\frac{dR}{dz}\right) = 0$$

$$\frac{r'^2 d\lambda'^2 + dr'^2(1+s^2) + 2r's\,dr'\,ds + r'^2 ds^2}{dt^2} - \frac{2\mu}{r'(1+s^2)^{\frac{1}{2}}} + \frac{\mu}{a} + 2\int dR = 0$$

dR being the differential of R with regard only to the co-ordinates of the planet m.

$$\frac{d^2r'(1+s^2) - r'\,d\lambda'^2 + 2s\,dr'\,ds + r's\,d^2s}{dt^2} + \frac{\mu}{r'(1+s^2)^{\frac{1}{2}}} + \left(\frac{dR}{dr'}\right) = 0$$

$$\text{d.}\frac{r'^2 d\lambda'}{dt} + \left(\frac{dR}{d\lambda'}\right) = 0$$

$$\frac{r's\,d^2r' + 2r'\,dr'\,ds + r'^2 d^2s}{dt^2} + \frac{s}{r'(1+s^2)^{\frac{1}{2}}} + \left(\frac{dR}{ds}\right) = 0$$

$$\frac{r'\mathrm{d}^2r'-r'^2\mathrm{d}\lambda'^2}{\mathrm{d}t^2}+\frac{\mu}{r'^2(1+s^2)^{\frac{1}{2}}}+r'\left(\frac{\mathrm{d}R}{\mathrm{d}r'}\right)-s\left(\frac{\mathrm{d}R}{\mathrm{d}s}\right)=0$$

$$\frac{r'^2\mathrm{d}^2s+2r'\mathrm{d}r'\mathrm{d}s-r'^2s\mathrm{d}\lambda'^2}{\mathrm{d}t^2}+(1+s^2)\left(\frac{\mathrm{d}R}{\mathrm{d}s}\right)-r's\left(\frac{\mathrm{d}R}{\mathrm{d}r'}\right)=0$$

$$\frac{\mathrm{d}^2.r'^2(1+s^2)}{2\mathrm{d}t^2}-\frac{\mu}{r'(1+s^2)^{\frac{1}{2}}}+\frac{\mu}{a}+2\int\mathrm{d}R+r'\left(\frac{\mathrm{d}R}{\mathrm{d}r'}\right)=0$$

Making λ the independent variable instead of t,

$$\frac{\mathrm{d}^2r'}{\mathrm{d}t^2}-2\frac{\mathrm{d}r'\mathrm{d}^2t}{\mathrm{d}t^3}-r'^2\frac{\mathrm{d}\lambda'^2}{\mathrm{d}t^2}+\frac{\mu}{r'^2(1+s^2)^{\frac{1}{2}}}+r'\left(\frac{\mathrm{d}R}{\mathrm{d}r'}\right)-s\left(\frac{\mathrm{d}R}{\mathrm{d}s}\right)=0$$

$$r'^4\frac{\mathrm{d}\lambda'^2}{\mathrm{d}t^2}=h^2-2\int r'^2\left(\frac{\mathrm{d}R}{\mathrm{d}\lambda'}\right)\mathrm{d}\lambda',\ h \text{ being a constant,}$$

$$4r'^3dr'\frac{\mathrm{d}\lambda'^2}{\mathrm{d}t^2}-2r'^4\frac{\mathrm{d}\lambda'^2\mathrm{d}^2t}{\mathrm{d}t^3}=-2r'^2\left(\frac{\mathrm{d}R}{\mathrm{d}\lambda'}\right)\mathrm{d}\lambda'$$

$$\left\{\frac{\mathrm{d}^2.\frac{1}{r'}}{\mathrm{d}\lambda'^2}+\frac{1}{r'}\right\}\left\{1-\frac{2}{h^2}\int r'^2\left(\frac{\mathrm{d}R}{\mathrm{d}\lambda'}\right)\mathrm{d}\lambda'\right\}-\frac{\mu}{h^2(1+s^2)^{\frac{1}{2}}}$$

$$-\frac{r'}{h^2}\left\{r'\left(\frac{\mathrm{d}R}{\mathrm{d}r'}\right)-s\left(\frac{\mathrm{d}R}{\mathrm{d}s}\right)-\frac{1}{r'}\left(\frac{\mathrm{d}R}{\mathrm{d}\lambda'}\right)\frac{\mathrm{d}r'}{\mathrm{d}\lambda'}\right\}=0$$

$$\left\{\frac{\mathrm{d}^2s}{\mathrm{d}\lambda'^2}+s\right\}\left\{1-\frac{2}{h^2}\int r'^2\left(\frac{\mathrm{d}R}{\mathrm{d}\lambda'}\right)\mathrm{d}\lambda'\right\}$$

$$+\frac{r'^2}{h^2}\left\{(1+s^2)\left(\frac{\mathrm{d}R}{\mathrm{d}s}\right)-r'\left(\frac{\mathrm{d}R}{\mathrm{d}r'}\right)-\left(\frac{\mathrm{d}R}{\mathrm{d}\lambda'}\right)\frac{\mathrm{d}s}{\mathrm{d}\lambda'}\right\}=0$$

When the disturbing force is neglected

$$\frac{\mathrm{d}.r'^2\mathrm{d}\lambda'}{\mathrm{d}t}=0,\quad\frac{\mathrm{d}^2.\frac{1}{r'}}{\mathrm{d}\lambda'^2}+\frac{1}{r'}-\frac{\mu}{h^2(1+s^2)^{\frac{1}{2}}}=0,\quad\frac{\mathrm{d}^2s}{\mathrm{d}\lambda'^2}+s=0$$

of which equations the integrals are,

$$r'^2\,\mathrm{d}\lambda'=h\,\mathrm{d}t,\quad\frac{1}{r'}=\frac{\mu\cos\iota^2}{h^2}\left\{(1+s^2)^{\frac{1}{2}}+e\cos(\lambda'-\varpi)\right\}$$

$$s=\tan\iota\sin(\lambda'-\nu).$$

If $\mathrm{d}t=\sqrt{\frac{a}{\mu}}r'\,\mathrm{d}v$, and v be taken for the independent variable

$$r'\frac{\mathrm{d}^2r'}{\mathrm{d}t^2}-r'\,\mathrm{d}r'\frac{\mathrm{d}^2t}{\mathrm{d}t^3}-r'^2\frac{\mathrm{d}\lambda'^2}{\mathrm{d}t^2}+\frac{\mu}{r'(1+s^2)^{\frac{1}{2}}}+r'\left(\frac{\mathrm{d}R}{\mathrm{d}r'}\right)-s\left(\frac{\mathrm{d}R}{\mathrm{d}s}\right)=0$$

$$d^2 t = \left(\frac{a}{\mu}\right)^{\frac{1}{2}} dr' dv$$

$$\frac{d^2 r'}{dv^2} + \left(\frac{d.r's}{dv}\right)^2 - \frac{a}{(1+s^2)^{\frac{1}{2}}} + r' + \frac{ar'}{\mu}\left\{2\int dR + r'\left(\frac{dR}{dr'}\right) - s\left(\frac{dR}{ds}\right)\right\} = 0.$$

If the orbit of the planet m coincide with the plane $x\,y$, s is of the order of the disturbing force, of which therefore neglecting the square, $r' = r$, $\lambda' = \lambda$

$$\frac{d^2 r}{dv^2} - a + r + \frac{ar}{\mu}\left\{2\int dR + r\left(\frac{dR}{dr}\right)\right\} = 0.$$

When the disturbing force is neglected, the integral of this equation is $r = a\{1 - e\cos(v-\alpha)\}$, v being the eccentric anomaly.

$$nt + \varepsilon - \varpi = v - \alpha - e\sin(v-\alpha)$$

$$\tan\frac{\lambda-\varpi}{2} = \left\{\frac{1+e}{1-e}\right\}^{\frac{1}{2}}\tan\frac{v-\alpha}{2}.$$

If Q be put for the quantity $\frac{ar}{\mu}\left\{2\int dR + r\left(\frac{dR}{dr}\right)\right\}$ and the constant α, which may afterwards be replaced, be omitted for the present,

$$r = a\{1 - e\cos v\} - \sin v\int Q\cos v\,dv + \cos v\int Q\sin v\,dv$$

$$dt = \sqrt{\frac{a}{\mu}}\left\{a\{1-e\cos v\} - \sin v\int Q\cos v\,dv + \cos v\int Q\sin v\,dv\right\}dv$$

$$nt + \varepsilon - \varpi = v - e\sin v - \frac{1}{a}\left\{\int Q\,dv - \cos v\int Q\cos v\,dv - \sin v\int Q\sin v\,dv\right\}$$

If $v = f(nt + \varepsilon - \varpi)$ in the elliptic theory, then neglecting the square of the disturbing force,

$$v = f(nt+\varepsilon-\varpi) + \frac{d.f(nt+\varepsilon-\varpi)}{dt}\frac{1}{a}\left\{\int Q\,dv - \cos v\int Q\cos v\,dv - \sin v\int Q\sin v\right\}dv$$

If δv, δr denote the values of those parts of r and v which are due to the first power of the disturbing force,

$$\delta v = \frac{dv}{andt}\left\{\int Q\,dv - \cos v\int Q\cos v\,dv - \sin v\int Q\sin v\,dv\right\}$$

$$\delta r = ae\sin v\,\delta v - \sin v\int Q\cos v\,dv + \cos v\int Q\sin v\,dv$$

$$= \frac{\sin v}{1-e\cos v}\int Q\{e - \cos v\}\,dv - \frac{e-\cos v}{1-e\cos v}\int Q\sin v\,dv.$$

In the elliptic theory,

$$dv = \frac{n\,dt}{1-e\cos v}, \qquad \frac{\cos v - e}{1-e\cos v} = \cos\lambda, \qquad \frac{\sin v(1-e^2)^{\frac{1}{2}}}{1-e\cos v} = \sin\lambda$$

therefore,

$$\delta r = \frac{a n \cos\lambda \int r \sin\lambda \left\{ 2\int \mathrm{d}R + r\left(\frac{\mathrm{d}R}{\mathrm{d}r}\right)\right\}\mathrm{d}t - a n \sin\lambda \int r\cos\lambda \left\{ 2\int \mathrm{d}R + r\left(\frac{\mathrm{d}R}{\mathrm{d}r}\right)\right\}\mathrm{d}t}{\mu(1-e^2)^{\frac{1}{2}}}$$

which is the equation X of the Mécanique Céleste, vol. i. p. 258.

Multiplying the equation of p. 114, l. 2, by $\frac{2r'^2\mathrm{d}\lambda'}{\mathrm{d}t}$, and integrating, $\frac{r'^4\mathrm{d}\lambda'^2}{\mathrm{d}t^2}$ $+2\int r'^2\left(\frac{\mathrm{d}R}{\mathrm{d}\lambda'}\right)\mathrm{d}\lambda' = h_0^2$, if $h^2 = h_0^2 - 2\int r'^2\left(\frac{\mathrm{d}R}{\mathrm{d}\lambda'}\right)\mathrm{d}\lambda'$, h being variable, and h_0 the value of h at a given epoch, $h\,\mathrm{d}h = -r'^2\left(\frac{\mathrm{d}R}{\mathrm{d}\lambda'}\right)\mathrm{d}\lambda'$, $r'^2\,\mathrm{d}\lambda' = h\,\mathrm{d}t$, and making λ' the independent variable instead of t, $2r'\,\mathrm{d}r'\,\mathrm{d}\lambda' = \mathrm{d}h\,\mathrm{d}t + h\,\mathrm{d}^2t$

$$r'\frac{\mathrm{d}^2r'}{\mathrm{d}t^2} - \frac{r'\mathrm{d}r'\mathrm{d}^2t}{\mathrm{d}t^3} - \frac{r'^4\mathrm{d}\lambda'^2}{\mathrm{d}t^2} + \frac{\mu}{r'(1+s^2)^{\frac{1}{2}}} + r'\left(\frac{\mathrm{d}R}{\mathrm{d}r'}\right) - s\left(\frac{\mathrm{d}R}{\mathrm{d}s}\right) = 0$$

$$r'\left(\frac{\mathrm{d}^2r'}{\mathrm{d}t^2}\right) + \frac{r'\mathrm{d}r'\mathrm{d}t}{h\,\mathrm{d}t^2} - \frac{2r'^3\mathrm{d}r'^2\mathrm{d}\lambda}{h\,\mathrm{d}t^2} - \frac{r'^4\mathrm{d}\lambda'^2}{\mathrm{d}t^2} + \frac{\mu}{r'(1+s^2)^{\frac{1}{2}}} + r'\left(\frac{\mathrm{d}R}{\mathrm{d}r'}\right) - s\left(\frac{\mathrm{d}R}{\mathrm{d}s}\right) = 0$$

$$\frac{\mathrm{d}^2.\frac{1}{r'}}{\mathrm{d}\lambda'^2} + \frac{1}{r'} - \frac{\mu}{h^2(1+s^2)^{\frac{1}{2}}} - \frac{r'}{h^2}\left\{r'\left(\frac{\mathrm{d}R}{\mathrm{d}r'}\right) - s\left(\frac{\mathrm{d}R}{\mathrm{d}s}\right) - \frac{1}{r'}\left(\frac{\mathrm{d}R}{\mathrm{d}\lambda}\right)\frac{\mathrm{d}r'}{\mathrm{d}\lambda'}\right\} = 0$$

$$\frac{\mathrm{d}^2s}{\mathrm{d}\lambda'^2} + s + \frac{r'^2}{h^2}\left\{(1+s^2)\left(\frac{\mathrm{d}R}{\mathrm{d}s}\right) - s\left(\frac{\mathrm{d}R}{\mathrm{d}s}\right) - \left(\frac{\mathrm{d}R}{\mathrm{d}\lambda}\right)\frac{\mathrm{d}s}{\mathrm{d}\lambda'}\right\} = 0$$

If all the constants in the elliptic integrals are supposed to vary, subject to the condition that they still satisfy these differential equations, and that the form of the first differential coefficients $\frac{\mathrm{d}r'}{\mathrm{d}\lambda'}$, $\frac{\mathrm{d}s}{\mathrm{d}\lambda'}$, remains unaltered,

$$\frac{\mathrm{d}.\frac{1}{r'}}{\mathrm{d}\lambda'} = \frac{\mu\cos\imath^2}{h^3}\left\{\frac{s}{(1+s^2)^{\frac{1}{2}}}\frac{\mathrm{d}s}{\mathrm{d}\lambda'} - e\sin(\lambda'-\varpi)\right\}, \qquad \frac{\mathrm{d}s}{\mathrm{d}\lambda'} = \tan\imath\cos(\lambda'-\nu)$$

$$-2\{(1+s^2)^{\frac{1}{2}} + e\cos(\lambda'-\varpi)\}\{h^2\sin\imath\,\mathrm{d}\imath + \cos\imath\,h\,\mathrm{d}h\}$$
$$+h^2\cos\imath\cos(\lambda'-\varpi)\,\mathrm{d}e + h^2\cos\imath\,e\sin(\lambda'-\varpi)\,\mathrm{d}\varpi = 0$$

$$-2\left\{\frac{s\tan\imath\cos(\lambda'-\nu)}{(1+s^2)^{\frac{1}{2}}} - e\sin(\lambda'-\varpi)\right\}\{h^2\sin\imath\,\mathrm{d}\imath + \cos\imath\,h\,\mathrm{d}h\}$$
$$-h^2\cos\imath\sin(\lambda'-\varpi)\,\mathrm{d}e + h^2\cos\imath\,e\cos(\lambda'-\varpi)\,\mathrm{d}\varpi$$
$$-\frac{\cos\imath\,s\,r'^2}{(1+s^2)^{\frac{1}{2}}}\left\{(1+s^2)\left(\frac{\mathrm{d}R}{\mathrm{d}s}\right) - r'\left(\frac{\mathrm{d}R}{\mathrm{d}r'}\right) - \left(\frac{\mathrm{d}R}{\mathrm{d}\lambda}\right)\left(\frac{\mathrm{d}s}{\mathrm{d}\lambda}\right)\right\}\mathrm{d}\lambda'$$

$$-\frac{h^2 r'}{\mu \cos i}\left\{r'\left(\frac{dR}{dr'}\right)-s\left(\frac{dR}{ds}\right)-\frac{1}{r'}\left(\frac{dR}{d\lambda'}\right)\left(\frac{dr'}{d\lambda'}\right)\right\}d\lambda'=0$$

$$\sin(\lambda'-\nu)\frac{di}{\cos i^2}-\tan i\cos(\lambda'-\nu)\,d\nu=0$$

$$\cos(\lambda'-\nu)\frac{di}{\cos i^2}+\tan i\sin(\lambda'-\nu)\,d\nu+\frac{r'^2}{h^2}\left\{(1+s^2)\left(\frac{dR}{ds}\right)-s\left(\frac{dR}{dr'}\right)-\left(\frac{dR}{d\lambda'}\right)\left(\frac{ds}{d\lambda'}\right)\right\}d\lambda'=0$$

$$h\,dh=-r'^2\left(\frac{dR}{d\lambda'}\right)d\lambda'$$

Whence by elimination,

$$h^2\cos i\,de+2\left\{(1+s^2)^{\frac{1}{2}}\cos(\lambda'-\varpi)+e-\frac{\tan i\,s\cos(\lambda'-\nu)\sin(\lambda'-\varpi)}{(1+s^2)^{\frac{1}{2}}}\right\}$$

$$\left\{\sin i\cos^2 i\,r'^2\cos(\lambda'-\nu)\left\{(1+s^2)\left(\frac{dR}{ds}\right)-s\left(\frac{dR}{dr'}\right)-\left(\frac{dR}{d\lambda'}\right)\left(\frac{ds}{d\lambda'}\right)\right\}+\cos i\,r'^2\left(\frac{dR}{d\lambda'}\right)\right\}d\lambda'$$

$$+\frac{s\cos i\,r'^2\sin(\lambda'-\varpi)}{(1+s^2)^{\frac{1}{2}}}\left\{(1+s^2)\left(\frac{dR}{ds}\right)-r'\left(\frac{dR}{dr'}\right)-\left(\frac{dR}{d\lambda'}\right)\left(\frac{ds}{d\lambda'}\right)\right\}d\lambda'$$

$$+\frac{h^2 r'}{\mu\cos i}\sin(\lambda'-\varpi)\left\{r'\left(\frac{dR}{dr'}\right)-s\left(\frac{dR}{ds}\right)-\frac{1}{r'}\left(\frac{dR}{d\lambda'}\right)\left(\frac{dr'}{d\lambda'}\right)\right\}d\lambda'=0$$

$$h^2\cos i\,e\,d\varpi+2\left\{(1+s^2)^{\frac{1}{2}}\sin(\lambda'-\varpi)+\frac{s\tan i\cos(\lambda'-\nu)\cos(\lambda'-\varpi)}{(1+s^2)^{\frac{1}{2}}}\right\}$$

$$\left\{\sin i\cos^2 i\,r'^2\cos(\lambda'-\nu)\left\{(1+s^2)\left(\frac{dR}{ds}\right)-s\left(\frac{dR}{dr'}\right)-\left(\frac{dR}{d\lambda'}\right)\left(\frac{ds}{d\lambda'}\right)\right\}+\cos i\,r'^2\left(\frac{dR}{d\lambda'}\right)\right\}d\lambda'$$

$$-\frac{s\cos i\,r'^2\cos(\lambda-\varpi)}{(1+s^2)^{\frac{1}{2}}}\left\{(1+s^2)\left(\frac{dR}{ds}\right)-r'\left(\frac{dR}{dr'}\right)-\left(\frac{dR}{d\lambda'}\right)\left(\frac{ds}{d\lambda'}\right)\right\}d\lambda'$$

$$-\frac{h^2 r'}{\mu\cos i}\cos(\lambda'-\varpi)\left\{r'\left(\frac{dR}{dr'}\right)-s\left(\frac{dR}{ds}\right)-\frac{1}{r'}\left(\frac{dR}{d\lambda'}\right)\left(\frac{dr'}{d\lambda'}\right)\right\}d\lambda'=0$$

$$di+\frac{r'^2\cos i^2\cos(\lambda'-\nu)}{h^2}\left\{(1+s^2)\left(\frac{dR}{ds}\right)-s\left(\frac{dR}{dr'}\right)-\left(\frac{dR}{d\lambda'}\right)\left(\frac{dr'}{d\lambda'}\right)\right\}d\lambda'=0$$

$$d\nu+\frac{r'^2\sin(\lambda'-\nu)}{h^2\tan i}\left\{(1+s^2)\left(\frac{dR}{ds}\right)-s\left(\frac{dR}{dr'}\right)-\left(\frac{dR}{d\lambda'}\right)\left(\frac{ds}{d\lambda'}\right)\right\}d\lambda'=0$$

If $x = \frac{r\cos\lambda'}{(1+s^2)^{\frac{1}{2}}}$, $y = \frac{r\sin\lambda'}{(1+s^2)^{\frac{1}{2}}}$, $z = \frac{rs}{(1+s^2)^{\frac{1}{2}}}$,

$$\frac{\mathrm{d}r^2 + \frac{r^2}{1+s^2}\left(\frac{\mathrm{d}s^2}{1+s^2} + \mathrm{d}\lambda'^2\right)}{\mathrm{d}t^2} - \frac{2\mu}{r} + \frac{\mu}{a} + 2\int \mathrm{d}R = 0$$

$$\frac{\mathrm{d}^2 r}{\mathrm{d}t^2} - \frac{r}{1+s^2}\frac{\left(\frac{\mathrm{d}s^2}{1+s^2} + \mathrm{d}\lambda'^2\right)}{\mathrm{d}t^2} + \frac{\mu}{r^2} + \left(\frac{\mathrm{d}R}{\mathrm{d}r}\right) = 0$$

$$\mathrm{d}.\frac{\frac{r^2}{1+s^2}\cdot\frac{\mathrm{d}\lambda'}{\mathrm{d}t}}{\mathrm{d}t} + \left(\frac{\mathrm{d}R}{\mathrm{d}\lambda'}\right) = 0$$

$$\frac{r^2\,\mathrm{d}^2 s + 2r\,\mathrm{d}r\,\mathrm{d}s - \frac{2r^2 s\,\mathrm{d}s^2}{(1+s^2)} + r^2 s\,\mathrm{d}\lambda'^2}{\mathrm{d}t^2} + (1+s^2)^2\left(\frac{\mathrm{d}R}{\mathrm{d}s}\right) = 0$$

Of which equations the integrals are

$$\frac{r^2}{1+s^2}.\,\mathrm{d}\lambda' = h\,\mathrm{d}t,\quad \frac{1}{r} = \frac{\mu\cos\iota^2}{h^2(1+s^2)^{\frac{1}{2}}}\left\{(1+s^2)^{\frac{1}{2}} + e\cos(\lambda' - \varpi)\right\}$$

$$s = \tan\iota\sin(\lambda' - \nu)$$

If $\mathrm{d}t = \sqrt{\frac{a}{\mu}}\,r\,\mathrm{d}v$, and v be taken for the independent variable

$$\frac{\mathrm{d}^2 r}{\mathrm{d}v^2} - a + r + \frac{ar}{\mu}\left\{2\int \mathrm{d}R + r\left(\frac{\mathrm{d}R}{\mathrm{d}r}\right)\right\} = 0$$

$r = a\{1 - e'\cos(v - \alpha)\}$ in the elliptic motion.

e' is accented for the present in order to distinguish it from e.

If the constants in the elliptic integrals are supposed to vary, subject to the condition that they still satisfy these differential equations, and that the form of the first differential coefficient $\frac{\mathrm{d}r}{\mathrm{d}v}$ remains unaltered,

$$\mathrm{d}^2 t = \frac{r\,\mathrm{d}v\,\mathrm{d}a}{2\sqrt{a\mu}} + \sqrt{\frac{a}{\mu}}\,\mathrm{d}r\,\mathrm{d}v,\quad \frac{r\,\mathrm{d}^2 r}{\mathrm{d}t^2} = \frac{r\,\mathrm{d}r^2}{\mathrm{d}t^2} - \frac{r\,\mathrm{d}r\,\mathrm{d}^2 t}{\mathrm{d}t^2}$$

$$\frac{\mathrm{d}^2 r}{\mathrm{d}v^2} - a + r + \frac{ar^2}{\mu}\left(\frac{\mathrm{d}R}{\mathrm{d}r}\right) - \frac{\mathrm{d}r\,\mathrm{d}a}{2a\,\mathrm{d}v^2} = 0$$

$$\left(1 - e'\cos(v-\alpha)\right)\mathrm{d}a - a\cos(v-\alpha)\,\mathrm{d}e' - ae'\sin(v-\alpha)\,\mathrm{d}\alpha = 0$$

$$e'\sin(v-\alpha)\,\mathrm{d}a + a\sin(v-\alpha)\,\mathrm{d}e' - ae'\cos(v-\alpha)\,\mathrm{d}\alpha$$

$$+\frac{a r^2}{\mu}\left(\frac{\mathrm{d}R}{\mathrm{d}r}\right)\mathrm{d}v+\frac{a^2 e}{\mu}\sin(v-\alpha)\,\mathrm{d}R=0$$

$$\mathrm{d}a=-\frac{2a^2}{\mu}\mathrm{d}R$$

$$\mathrm{d}e'=\frac{a}{\mu}\{2e'-2\cos(v-\alpha)-e'\sin(v-\alpha)^2\}\,\mathrm{d}R-\frac{r^2}{\mu}\sin(v-\alpha)\left(\frac{\mathrm{d}R}{\mathrm{d}r}\right)\mathrm{d}v$$

$$e\,\mathrm{d}\alpha=\frac{a}{\mu}\sin(v-\alpha)\{e'\cos(v-\alpha)-2\}\,\mathrm{d}R+\frac{r^2}{\mu}\cos(v-\alpha)\left(\frac{\mathrm{d}R}{\mathrm{d}r}\right)\mathrm{d}v$$

$$\int n\,\mathrm{d}t+\varepsilon-\varpi=v-\alpha-e'\sin(v-\alpha)$$

$$\mathrm{d}\varepsilon-\mathrm{d}\varpi=-\mathrm{d}\alpha-\sin(v-\alpha)\,\mathrm{d}e'+e'\cos(v-\alpha)\,\mathrm{d}\alpha.$$

The equation of condition which obtains between the constants a, e, ϖ, ν, ι and h may be found from the equation

$$\frac{\mathrm{d}r^2+\frac{r^2}{1+s^2}\left(\frac{\mathrm{d}s^2}{1+s^2}+\mathrm{d}\lambda^2\right)}{\mathrm{d}t^2}-\frac{2\mu}{r}+\frac{\mu}{a}=0$$

$$\frac{\mathrm{d}r^2}{\mathrm{d}t^2}+\frac{h^2}{r^2\cos\iota^2}-\frac{2\mu}{r}+\frac{\mu}{a}=0$$

which gives

$$\frac{\mu\cos\iota^2}{h^2}\left\{e^2\cos^2\iota\sin^2(\nu-\varpi)-\{1+e\cos(\nu-\varpi)\}\{1-e\cos(\nu-\varpi)\}\right\}+\frac{1}{a}=0$$

Equating the values of r which have been found,

$$\frac{h^2\sqrt{1+s^2}}{\mu\cos\iota^2\{\sqrt{1+s^2}+e\cos(\lambda-\varpi)\}}=a\{1-e'\cos(v-\alpha)\}$$

since the origin of the angle v is arbitrary, we may suppose $\lambda-\varpi=0$, and $v-\alpha=0$ at the same time,

so that $$\frac{h^2\sqrt{1+\tan\iota^2\sin(\varpi-\nu)^2}}{\mu\cos\iota^2\{\sqrt{1+\tan\iota^2\sin(\varpi-\nu)^2}+e\}}=a(1-e')$$

All the equations which have hitherto been proved are rigorously true, whatever powers of the disturbing function be retained. They are susceptible of simplification when the square and higher powers of the disturbing function are neglected: in this case, if the orbit be supposed to coincide with the plane $x\,y$, $\tan\iota=0$, and if the longitude be reckoned from the perihelion of the planet P,

$$h^2\,\mathrm{d}e+\{2\cos\lambda+e+e\cos\lambda^2\}r^2\left(\frac{\mathrm{d}R}{\mathrm{d}\lambda}\right)\mathrm{d}\lambda+\frac{h^2r^2\sin\lambda}{\mu}\left(\frac{\mathrm{d}R}{\mathrm{d}r}\right)\mathrm{d}\lambda=0$$

$$h^2 e\,d\varpi + \{2 + \cos\lambda\}\sin\lambda\, r^2\left(\frac{dR}{d\lambda}\right)d\lambda - h^2 r^2\cos\lambda\left(\frac{dR}{dr}\right)d\lambda = 0$$

$$d\iota + r^2\frac{\cos(\lambda-\nu)}{\mu}\left(\frac{dR}{ds}\right)d\lambda = 0$$

$$d\nu + r^2\frac{\sin(\lambda-\nu)}{\mu}\left(\frac{dR}{ds}\right)d\lambda = 0$$

by equation, p. 336, line 12, $\mu\, a\,(1-e^2) = h^2$, and by equation, line 17, $e = e'$.

$$de = \frac{a}{\mu}\left\{2e - 2\cos(v-\alpha) - e\sin(v-\alpha)^2\right\}dR - \frac{r^2}{\mu}\sin(v-\alpha)\left(\frac{dR}{dr}\right)dv$$

$$e\,d\alpha = \frac{a}{\mu}\left\{e\cos(v-\alpha) - 2\right\}\sin(v-\alpha)\,dR + \frac{r^2}{\mu}\cos(v-\alpha)\left(\frac{dR}{dr}\right)dv$$

$$\int n\,dt + s - \varpi = v - \alpha - e\sin(v-\alpha)$$

$$ds - d\varpi = -\{1 - e\cos(v-\alpha)\}\,d\alpha - \sin(v-\alpha)\,de$$

$$ds - d\varpi = -\frac{r^2}{a^2\sqrt{1-e^2}}d\varpi - \left\{\frac{r}{a(1-e^2)} + 1\right\}\sin(v-\alpha)\,de$$

$$\frac{\frac{h^2}{\mu}}{1 + e\cos(\lambda-\varpi)} = a\{1 - e\cos(v-\alpha)\}$$

$$de = -\frac{a\,n\,dt}{\mu\sqrt{1-e^2}}\left\{2\cos(\lambda-\varpi) + e + e\cos(\lambda-\varpi)^2\right\}\left(\frac{dR}{d\lambda}\right)$$
$$- \frac{a^2 n\,dt\sqrt{1-e^2}}{\mu}\sin(\lambda-\varpi)\left(\frac{dR}{dr}\right)$$

$$e\,d\varpi = -\frac{a\,n\,dt}{\mu\sqrt{1-e^2}}\left\{2 + e\cos(\lambda-\varpi)\right\}\sin(\lambda-\varpi)\left(\frac{dR}{d\lambda}\right)$$
$$+ \frac{a^2 n\,dt\sqrt{1-e^2}}{\mu}\cos(\lambda-\varpi)\left(\frac{dR}{dr}\right)$$

$$ds - d\varpi = -\sqrt{1-e^2}\,d\varpi + \frac{2a^2 n(1-e^2)\,dt}{\mu\{1 + e\cos(\lambda-\varpi)\}}\left(\frac{dR}{dr}\right)$$

If the longitudes be reckoned from the perihelion of the planet P,

$$\frac{de}{dt} = -\frac{a\,n(\cos\lambda + e)}{\mu\sqrt{1-e^2}}\left(\frac{dR}{d\lambda}\right) - \frac{m_1 a^2 n\sqrt{1-e^2}}{\mu}\left\{\frac{r_1'\sin\lambda_1'}{r_1'^3} + \frac{r\sin\lambda - r_1'\sin\lambda_1'}{\{r^2 - 2rr_1'\cos(\lambda-\lambda_1') + r_1'^2\}^{\frac{3}{2}}}\right\}$$

$$\frac{\mathrm{d}\varpi}{\mathrm{d}t} = -\frac{an\sin\lambda}{\mu\sqrt{1-e^2}}\left(\frac{\mathrm{d}R}{\mathrm{d}\lambda}\right) - \frac{m_1 a^2 n\sqrt{1-e^2}}{\mu}\left\{\frac{r_1'\cos\lambda_1'}{r_1'^3} + \frac{r\cos\lambda - r_1'\cos\lambda_1'}{\{r^2 - 2rr_1'\cos(\lambda-\lambda_1') + r_1'^2\}^{\frac{3}{2}}}\right\}$$

and since $n\,\mathrm{d}t = \frac{r}{a}\,\mathrm{d}v$,

$$\frac{\mathrm{d}e}{\mathrm{d}v} = -\frac{r(\cos\lambda + e)}{\mu\sqrt{1-e^2}}\left(\frac{\mathrm{d}R}{\mathrm{d}\lambda}\right) - \frac{m_1 ar\sqrt{1-e^2}}{\mu}\left\{\frac{r_1'\sin\lambda_1'}{r_1'^3} + \frac{r\sin\lambda - r_1'\sin\lambda_1'}{\{r^2 - 2rr_1'\cos(\lambda-\lambda_1') + r_1'^2\}^{\frac{3}{2}}}\right\}$$

$$\frac{e\,\mathrm{d}\varpi}{\mathrm{d}v} = -\frac{r\sin\lambda}{\mu\sqrt{1-e^2}}\left(\frac{\mathrm{d}R}{\mathrm{d}\lambda}\right) + \frac{m_1 ar\sqrt{1-e^2}}{\mu}\left\{\frac{r_1'\cos\lambda_1'}{r_1'^3} + \frac{r\cos\lambda - r_1'\cos\lambda_1'}{\{r^2 - 2rr_1'\cos(\lambda-\lambda_1') + r_1'^2\}^{\frac{3}{2}}}\right\}$$

$$\frac{\mathrm{d}\epsilon - \mathrm{d}\varpi}{\mathrm{d}v} = -\frac{r^2}{a^2\sqrt{1-e^2}}\frac{\mathrm{d}\varpi}{\mathrm{d}v} - \left\{\frac{r}{a(1-e^2)} + 1\right\}\sin v\,\frac{\mathrm{d}e}{\mathrm{d}v}$$

$$\mathrm{d}a = -\frac{2a^2}{\mu}\,\mathrm{d}R$$

and since $r^2\,\mathrm{d}\lambda = \sqrt{\mu a(1-e^2)}\,\mathrm{d}t = a\sqrt{1-e^2}\,r\,\mathrm{d}v$

$$\frac{\mathrm{d}i}{\mathrm{d}v} + \frac{a\sqrt{1-e^2}\,r\cos\lambda}{\mu}\left(\frac{\mathrm{d}R}{\mathrm{d}i}\right) = 0$$

$$\frac{\mathrm{d}\nu}{\mathrm{d}v} + \frac{a\sqrt{1-e^2}\,r\sin\lambda}{\mu}\left(\frac{\mathrm{d}R}{\mathrm{d}i}\right) = 0.$$

The last six equations serve to determine the perturbations of a comet.

Let $(\Delta e)_n$ be the variation of any element e during the variation Δv of v at any given epoch n, neglecting the square and higher powers of Δv,

$$(\Delta e)_n = \left(\frac{\mathrm{d}e}{\mathrm{d}v}\right)_n \Delta v$$

If the values of $(\Delta e)_n$ be calculated for the epochs $0, 1, 2 \ldots m$ corresponding to the values v, $v + i\,\Delta v$, $v + 2i\,\Delta v$, &c., differing from each other by $i\,\Delta v$, then the whole variation (δe) of v corresponding to the variation $i\,\Delta v$ of v

$$= i\left\{(\Delta e)_0 + (\Delta e)_1 \ldots\ldots + (\Delta e)_{m-1}\right\}$$

$$+ \frac{i-1}{2}\left\{(\Delta e)_m - (\Delta e)_0\right\} - \frac{(i-1)(i+1)}{12}\left\{(\Delta^2 e)_m - (\Delta^2 e)_0\right\} + \&c.$$

$$= i\left\{\left(\frac{\mathrm{d}e}{\mathrm{d}v}\right)_0 + \left(\frac{\mathrm{d}e}{\mathrm{d}v}\right)_1 \ldots\ldots + \left(\frac{\mathrm{d}e}{\mathrm{d}v}\right)_{m-1}\right\}\Delta v$$

$$+ \frac{i-1}{2}\left\{\left(\frac{\mathrm{d}e}{\mathrm{d}v}\right)_m - \left(\frac{\mathrm{d}e}{\mathrm{d}v}\right)_0\right\}\Delta v - \frac{(i-1)(i+1)}{12}\left\{\Delta\left(\frac{\mathrm{d}e}{\mathrm{d}v}\right)_m - \Delta\left(\frac{\mathrm{d}e}{\mathrm{d}v}\right)_0\right\}\Delta v.$$

When the interval Δv is indefinitely diminished, $i \Delta v$ is still equal to the variation of v between the epochs for which the quantities $\left(\frac{de}{dv}\right)_0$, $\left(\frac{de}{dv}\right)_1$ &c. are calculated, and

$$\delta e = i \Delta v \left\{ \left(\frac{de}{dv}\right)_0 + \left(\frac{de}{dv}\right)_1 \cdots\cdots + \left(\frac{de}{dv}\right)_{m-1} + \frac{1}{2}\left\{\left(\frac{de}{dv}\right)_m - \left(\frac{de}{dv}\right)_0\right\} - \frac{1}{12}\left\{\Delta\left(\frac{de}{dv}\right)_m - \Delta\left(\frac{de}{dv}\right)_0\right\} + \&c.\right\}$$

If the radius be taken for unity, and $i \Delta v$ is the mth part of the circumference; $i \Delta v = \frac{2 \times 3{\cdot}14159}{m}$, or, in other words, the resulting values of δe and δa in the equations given above must be multiplied by 2×3.14159, and divided by 360° expressed in the same unit as $i \Delta v$.

n is equal to the angular circumference divided by the periodic time expressed in the same unit as t; so that if a degree be taken as the unity of angular circumference, $n = 360°$ divided by the periodic time expressed in the same unit as t.

In the elliptic movement or first approximation

$\lambda = n t + \varepsilon +$ a series of sines of arcs multiples of $n t$ &c.

$\lambda_1 = n_1 t + \varepsilon_1 +$ a series of sines of arcs multiples of $n_1 t$ &c.

$\frac{a}{r} =$ constant $+$ a series of cosines.

$s =$ a series of sines.

$s_1 =$ a series of sines.

These values being substituted in the equations of p. 334 give $\frac{de}{d\lambda}$, $\frac{dk}{d\lambda}$, and $\frac{di}{d\lambda}$ each equal to a series of sines without any constant quantity, and $\frac{d\varpi}{d\lambda}$, $\frac{dv}{d\lambda}$ and $\frac{ds}{d\lambda}$ each equal to a series of cosines $+$ a constant quantity.

In the second approximation the values of λ, r and s retain the same form; and it is easy to see from the form of the expressions for $\frac{de}{d\lambda}$, $\frac{dk}{d\lambda}$, &c. p. 334, that the form of the values of these quantities is not altered however far the approximation be carried.

If the sun or primary be a spheroid, ϖ the angle which the plane of the sun's equator makes with the plane of the orbit of the planet; and if the longitude be reckoned from the line of intersection of the sun's equator with the orbit of the planet; R is increased by the quantity $c\left\{\frac{3\sin^2\varpi\sin^2\lambda-1}{r^3}\right\}$, c being a constant dependent upon the figure of the sun; but the partial differential coefficients of this quantity, which are introduced into the values of d e, d ϖ, &c. do not change the form of the expressions for those quantities.

If the planet move in a medium which resists according to any power n of the velocity, if c be a constant and v the velocity, the term $2c\int v^{n+1}\,\mathrm{d}\,t$ must be added to $2\int \mathrm{d}\,R$,

$$cv^{n-1}\left\{(1+s^2)\frac{\mathrm{d}\,r}{\mathrm{d}\,t}+r\,s\frac{\mathrm{d}\,s}{\mathrm{d}\,t}\right\} \text{ to } \left(\frac{\mathrm{d}\,R}{\mathrm{d}\,r}\right),$$

$$cv^{n-1}r^2\frac{\mathrm{d}\,\lambda}{\mathrm{d}\,t} \text{ to } \left(\frac{\mathrm{d}\,R}{\mathrm{d}\,\lambda}\right), \quad \text{and } cv^{n-1}r\frac{\mathrm{d}.r\,s}{\mathrm{d}\,t} \text{ to } \left(\frac{\mathrm{d}\,R}{\mathrm{d}\,s}\right)$$

in the equations of p. 330.

If the orbit of the planet be supposed to coincide with the plane $x\,y$, so that $s=0$, then by the equations of p. 337 after reductions

$$\mathrm{d}\,a=-2c\,a^2\left(\frac{\mu}{a}\right)^{\frac{n+1}{2}}\left\{\frac{1+e\cos v}{1-e\cos v}\right\}^{\frac{n+1}{2}}\frac{(1-e\cos v)}{n}\,\mathrm{d}\,v$$

$$\mathrm{d}\,e=-2c\left(\frac{\mu}{a}\right)^{\frac{n-1}{2}}\left\{\frac{1+e\cos v}{1-e\cos v}\right\}^{\frac{n-1}{2}}\left(\frac{1-e^2}{n}\right)\cos v\,\mathrm{d}\,v$$

$$e\,\mathrm{d}\,\varpi=-2c\left(\frac{\mu}{a}\right)^{\frac{n-1}{2}}\left\{\frac{1+e\cos v}{1-e\cos v}\right\}^{\frac{n-1}{2}}\frac{\sqrt{1-e^2}}{n}\sin v\,\mathrm{d}\,v$$

$$\mathrm{d}\,\varepsilon-\mathrm{d}\,\varpi=2c\left(\frac{\mu}{a}\right)^{\frac{n-1}{2}}\left\{\frac{1+e\cos v}{1-e\cos v}\right\}^{\frac{n-1}{2}}\frac{\sin v}{n}\left\{\frac{1-e^2\cos v}{e}\right\}\mathrm{d}\,v$$

The form of these equations differs from that which obtained before, now the variations of e, ϖ and ε are periodical, while that of a has a term which varies with the time. $\frac{\mathrm{d}\,e}{\mathrm{d}\,v}$ contains only odd powers of cos v and for that reason has no constant term. The periods of the periodic inequalities of all the elliptic constants due to the action of the resisting medium are fractional parts of the periodic time of the planet.

If the origin of t coincides with the instant of the perihelion passage, by LAGRANGE's theorem

$$\cos v = \cos nt - e\sin nt^2 - \frac{e^2}{2}\cdot\frac{\mathrm{d}.\sin nt^2}{\mathrm{d}.nt} - \frac{e^3}{2.3}\cdot\frac{\mathrm{d}^2.\sin nt^3}{(\mathrm{d}.nt)^2} - \frac{e^4}{2.3.4}\frac{\mathrm{d}^3\sin nt^4}{(\mathrm{d}.nt^3} - \&c.$$

$$\sin v = \sin nt + e\sin nt\cos nt + \frac{e^2}{2}\frac{\mathrm{d}.\sin nt^2\cos nt}{\mathrm{d}.nt} + \frac{e^3}{2.3}\frac{\mathrm{d}^2.\sin nt^3\cos nt}{(\mathrm{d}.nt)^2} + \frac{e^4}{2.3.4}\cdot\frac{\mathrm{d}^3.\sin nt^4\cos nt}{(\mathrm{d}.nt)^3}$$

$$\sin nt^2 = \frac{1-\cos 2nt}{2},\quad \sin nt^3 = \frac{3\sin nt - \sin 3nt}{4}$$

$$\sin nt^4 = \frac{3-4\cos 2nt+\cos 4nt}{8},\quad \sin nt^5 = \frac{10\sin nt - 5\sin 3nt + \sin 5nt}{16}$$

$$\frac{\mathrm{d}.\sin nt^3}{\mathrm{d}.nt} = \frac{3\cos nt - 3\cos 3nt}{4}$$

$$\frac{\mathrm{d}.\sin nt^4}{\mathrm{d}.nt} = \frac{2\sin 2nt - \sin 4nt}{2},\quad \frac{\mathrm{d}^2.\sin nt^4}{(\mathrm{d}.nt)^2} = 2\cos 2nt - 2\cos 4nt$$

$$\frac{\mathrm{d}.\sin nt^5}{\mathrm{d}.nt} = \frac{10\cos nt - 15\cos 3nt + 5\cos 5nt}{16}$$

$$\frac{\mathrm{d}^2\sin nt^5}{(\mathrm{d}.nt)^2} = \frac{-10\sin nt + 45\sin 3nt - 25\sin 5nt}{16}$$

$$\frac{\mathrm{d}^3.\sin nt^5}{(\mathrm{d}.nt)} = \frac{-10\cos nt + 135\cos 3nt - 125\cos 5nt}{16}$$

$$\cos v = \cos nt - \frac{e}{2} + \frac{e}{2}\cos 2nt - \frac{e^2}{2}\left\{\frac{3\cos nt - 3\cos 3nt}{4}\right\} - \frac{e^3}{2.3}\left\{2\cos 2nt - 2\cos 4nt\right\}$$
$$- \frac{e^4}{2.3.4}\left\{\frac{-10\cos nt + 135\cos 3nt - 125\cos 5nt}{16}\right\}$$

If the origin of the time does not coincide with the perihelion passage, $nt + \varepsilon - \varpi$ must be substituted for nt, but as ε always accompanies nt, it may be suppressed at present for convenience, and afterwards replaced.

$$\cos v = \left\{1 - \frac{3}{8}e^2 + \frac{5e^4}{192}\right\}\cos(nt-\varpi) - \frac{e}{2} + \frac{e}{2}\left\{1 - \frac{2e^2}{3}\right\}\cos(2nt - 2\varpi)$$
$$+ \frac{3}{8}e^2\left\{1 - \frac{15}{16}e^2\right\}\cos(3nt - 3\varpi) + \frac{e^3}{3}\cos(4nt - 4\varpi) + \frac{125}{384}e^4\cos(5nt - 5\varpi)$$

$$\sin nt^2\cos nt = \frac{\mathrm{d}.\sin nt^3}{3\,\mathrm{d}.nt} = \frac{\cos nt - \cos 3nt}{4},\quad \frac{\mathrm{d}.\sin nt^2\cos nt}{\mathrm{d}.nt} = \frac{-\sin nt + 3\sin 3nt}{4}$$

$$\sin nt^3\cos nt = \frac{\mathrm{d}.\sin nt^4}{4\,\mathrm{d}.nt} = \frac{2\sin 2nt - \sin 4nt}{8}, \qquad \frac{\mathrm{d}.\sin nt^4\cos nt}{\mathrm{d}.nt} = \frac{\cos 2nt - \cos 4nt}{2}$$

$$\frac{\mathrm{d}^2.\sin nt^4\cos nt}{(\mathrm{d}.nt)^2} = -\sin 2nt + 2\sin 4nt$$

$$\sin nt^4\cos nt = \frac{\mathrm{d}.\sin nt^5}{5\,\mathrm{d}.nt} = \frac{2\cos nt - 3\cos 3nt + \cos 5nt}{16}$$

$$\frac{\mathrm{d}.\sin nt^5\cos nt}{\mathrm{d}.nt} = \frac{-2\sin nt + 9\sin 3nt - 5\sin 5nt}{16}$$

$$\frac{\mathrm{d}^2.\sin nt^5\cos nt}{(\mathrm{d}.nt)^2} = \frac{-2\cos nt + 27\cos 3nt - 25\cos 5nt}{16}$$

$$\frac{\mathrm{d}^3.\sin nt^5\cos nt}{(\mathrm{d}.nt)^3} = \frac{2\sin nt - 81\sin 3nt + 125\sin 5nt}{16}$$

$$\sin v = \sin nt + \frac{e}{2}\sin 2nt + \frac{e^2}{2}\left\{\frac{-\sin nt + 3\sin 3nt}{4}\right\} + \frac{e^3}{2.3}\{-\sin 2nt + 2\sin 4nt\}$$
$$+ \frac{e^4}{2.3.4}\left\{\frac{2\sin nt - 81\sin 3nt + 125\sin 5nt}{16}\right\}$$

and replacing nt by $nt - \varpi$

$$\sin v = \left\{1 - \frac{3e^2}{8} + \frac{5}{192}e^4\right\}\sin(nt-\varpi) + \frac{e}{2}\left\{1 - 2\frac{e^2}{3}\right\}\sin(2nt - 2\varpi)$$
$$+ \left\{\frac{2e^2}{8} - \frac{4}{192}e^4\right\}\sin(nt-\varpi) + \frac{e^3}{6}\sin(2nt - 2\varpi)$$
$$+ \frac{3e^2}{8}\left\{1 - \frac{15}{16}e^2\right\}\sin(3nt - 3\varpi) + \frac{e^3}{3}\sin(4nt - 4\varpi) + \frac{125}{384}e^4\sin(5nt - 5\varpi)$$
$$+ \frac{3}{8}\cdot\frac{6}{16}e^4\sin(3nt - 3\varpi)$$

$$\cos(v+\varpi) = \left\{1 - \frac{e^2}{4} + \frac{e^4}{64}\right\}\cos nt - \frac{e}{2}\cos\varpi + \frac{e}{2}\left\{1 - \frac{e^2}{2}\right\}\cos(2nt - \varpi)$$
$$+ \frac{3}{8}e^2\left\{1 - \frac{3}{4}e^2\right\}\cos(3nt - 2\varpi) + \frac{e^3}{3}\cos(4nt - 3\varpi) + \frac{125}{384}e^4\cos(5nt - 4\varpi)$$
$$- \frac{e^2}{8}\left\{1 - \frac{e^2}{12}\right\}\cos(nt - 2\varpi) - \frac{e^3}{12}\cos(2nt - 3\varpi) - \frac{9}{128}e^4\cos(3nt - 4\varpi)$$

$$\sin(v+\varpi) = \left\{1 - \frac{e^2}{4} + \frac{e^4}{64}\right\}\sin nt - \frac{e}{2}\sin\varpi + \frac{e}{2}\left\{1 - \frac{e^2}{2}\right\}\sin(2nt - \varpi)$$
$$+ \frac{3}{8}e^2\left\{1 - \frac{3}{4}e^2\right\}\sin(3nt - 2\varpi) + \frac{e^3}{3}\sin(4nt - 3\varpi) + \frac{125}{384}e^4\sin(5nt - 4\varpi)$$

$$+\frac{e^3}{8}\left(1-\frac{e^2}{12}\right)\sin(nt-2\varpi)+\frac{e^3}{12}\sin(2nt-3\varpi)+\frac{9}{128}e^4\sin(3nt-4\varpi)$$

$$\cos(\upsilon-\varpi)=\left(1-\frac{e^2}{4}\right)\cos(nt-2\varpi)-\frac{e}{2}\cos\varpi+\frac{e}{2}\cos(2nt-3\varpi)+\frac{3}{8}e^2\cos(3nt-4\varpi)-\frac{e^3}{8}\cos nt$$

$$\sin(\upsilon-\varpi)=\left(1-\frac{e^2}{4}\right)\sin(nt-2\varpi)+\frac{e}{2}\sin\varpi+\frac{e}{2}\sin(2nt-3\varpi)+\frac{3}{8}e^2\sin(3nt-4\varpi)+\frac{e^3}{8}\sin nt$$

$$r\cos\lambda=r\cos(\lambda-\varpi)\cos\varpi-r\sin(\lambda-\varpi)\sin\varpi=a\{(\cos\upsilon-e)\cos\varpi-(1-e^2)^{\frac{1}{2}}\sin\upsilon\sin\varpi\}$$

$$r\sin\lambda=r\sin(\lambda-\varpi)\cos\varpi+r\cos(\lambda-\varpi)\sin\varpi=a\{(1-e^2)^{\frac{1}{2}}\sin\upsilon\cos\varpi+(\cos\upsilon-e)\sin\varpi\}$$

$$r\cos\lambda=a\left\{\left(1-\frac{e^2}{4}-\frac{e^4}{16}\right)\cos(\upsilon+\varpi)-e\cos\varpi+\frac{e^2}{4}\left(1+\frac{e^2}{4}\right)\cos(\upsilon-\varpi)\right\}$$

$$r\sin\lambda=a\left\{\left(1-\frac{e^2}{4}-\frac{e^4}{16}\right)\sin(\upsilon+\varpi)-e\sin\varpi-\frac{e^2}{4}\left(1+\frac{e^2}{4}\right)\sin(\upsilon-\varpi)\right\}$$

$$\begin{matrix}r\cos\lambda\\ r\sin\lambda\end{matrix}=a\left\{\left(1-\frac{e^2}{2}-\frac{e^4}{64}\right)\begin{matrix}\cos\\ \sin\end{matrix}nt-\frac{3}{2}e\begin{matrix}\cos\\ \sin\end{matrix}\varpi+\frac{e}{2}\left(1-\frac{3}{4}e^2\right)\begin{matrix}\cos\\ \sin\end{matrix}(2nt-\varpi)\right.$$
$$+\frac{3}{8}e^2\left(1-e^2\right)\begin{matrix}\cos\\ \sin\end{matrix}(3nt-2\varpi)+\frac{e^3}{3}\begin{matrix}\cos\\ \sin\end{matrix}(4nt-3\varpi)+\frac{125}{384}e^4\begin{matrix}\cos\\ \sin\end{matrix}(5nt-4\varpi)$$
$$\left.\begin{matrix}+\frac{e^3}{8}\left(1+\frac{e^2}{3}\right)\begin{matrix}\cos\\ \sin\end{matrix}(nt-2\varpi)+\frac{e^3}{24}\begin{matrix}\cos\\ \sin\end{matrix}(2nt-3\varpi)+\frac{3}{128}e^4\begin{matrix}\cos\\ \sin\end{matrix}(3nt-4\varpi)\&c.\\ -\end{matrix}\right\}$$

$$rr_i\begin{matrix}\cos\\ \sin\end{matrix}\{\lambda-\lambda_i\}=aa_i\left\{\left(1-\frac{e^2}{2}-\frac{e^4}{64}\right)\left(1-\frac{e_i^2}{2}-\frac{e_i^4}{64}\right)\begin{matrix}\cos\\ \sin\end{matrix}(nt-n_it)\right.$$
$$\begin{matrix}-\frac{3}{2}e\left(1-\frac{e_i^2}{2}\right)\begin{matrix}\cos\\ \sin\end{matrix}(n_it-\varpi)+\frac{e}{2}\left(1-\frac{3}{4}e^2\right)\left(1-\frac{e_i^2}{2}\right)\begin{matrix}\cos\\ \sin\end{matrix}(2nt-n_it-\varpi)\\ +\end{matrix}$$
$$+\frac{3}{8}e^2(1-e^2)\left(1-\frac{e_i^2}{2}\right)\begin{matrix}\cos\\ \sin\end{matrix}(3nt-n_it-2\varpi)+\frac{e^3}{3}\begin{matrix}\cos\\ \sin\end{matrix}(4nt-n_it-3\varpi)$$
$$+\frac{125}{384}e^4\begin{matrix}\cos\\ \sin\end{matrix}(5nt-n_it-4\varpi)\begin{matrix}+\frac{e^3}{8}\left(1+\frac{e^2}{3}\right)\left(1-\frac{e_i^2}{2}\right)\begin{matrix}\cos\\ \sin\end{matrix}(nt+n_it-2\varpi)\\ -\end{matrix}$$
$$\begin{matrix}+\frac{e^3}{24}\begin{matrix}\cos\\ \sin\end{matrix}(2nt+n_it-3\varpi)+\frac{3}{128}e^4\begin{matrix}\cos\\ \sin\end{matrix}(3nt+n_it-4\varpi)\\ -\end{matrix}$$
$$-\frac{3}{2}e_i\left(1-\frac{e^2}{2}\right)\begin{matrix}\cos\\ \sin\end{matrix}(nt-\varpi_i)+\frac{9}{4}ee_i\begin{matrix}\cos\\ \sin\end{matrix}(\varpi-\varpi_i)$$
$$-\frac{3}{4}ee_i\left(1-\frac{3}{4}e^2\right)\begin{matrix}\cos\\ \sin\end{matrix}(2nt-\varpi-\varpi_i)-\frac{9}{16}e^2e_i\begin{matrix}\cos\\ \sin\end{matrix}(3nt-2\varpi-\varpi_i)$$
$$\left.-\frac{e^2e_j}{2}\begin{matrix}\cos\\ \sin\end{matrix}(4nt-3\varpi-\varpi_i)\begin{matrix}-\frac{3}{16}e^2e_i\begin{matrix}\cos\\ \sin\end{matrix}(nt-2\varpi+\varpi_i)\\ +\end{matrix}\begin{matrix}-\frac{e^2e_j}{16}\begin{matrix}\cos\\ \sin\end{matrix}(2nt-3\varpi+\varpi_i)\\ +\end{matrix}\right.$$

$$+\frac{e_i}{2}\left(1-\frac{3}{4}e_i^2\right)\left(1-\frac{e^2}{2}\right)\begin{matrix}\cos\\ \sin\end{matrix}(nt-2n_it+\varpi_i)-\frac{3}{4}ee_i\left(1-\frac{3}{4}e_i^2\right)\begin{matrix}\cos\\ \sin\end{matrix}(2n_it-\varpi-\varpi_i)$$

$$+\frac{ee_i}{4}\left(1-\frac{3}{4}e^2\right)\left(1-\frac{3}{4}e_i^2\right)\begin{matrix}\cos\\ \sin\end{matrix}(2nt-2n_it-\varpi+\varpi_i)+\frac{3}{16}e^2e_i\begin{matrix}\cos\\ \sin\end{matrix}(3nt-2n_it-2\varpi+\varpi_i)$$

$$+\frac{e^2e_i}{6}\begin{matrix}\cos\\ \sin\end{matrix}(4nt-2n_it-3\varpi+\varpi_i)+\frac{e^2e_i}{16}\begin{matrix}\cos\\ \sin\end{matrix}(nt+2n_it-2\varpi-\varpi_i)$$

$$+\frac{e^2e_i}{48}\begin{matrix}\cos\\ \sin\end{matrix}(2nt+2n_it-3\varpi-\varpi_i)+\frac{3}{8}e_i^2(1-e_i^2)\left(1-\frac{e^2}{2}\right)\begin{matrix}\cos\\ \sin\end{matrix}(nt-3n_it+2\varpi_i)$$

$$-\frac{9}{16}ee_i^2\begin{matrix}\cos\\ \sin\end{matrix}(3n_it-\varpi-2\varpi_i)+\frac{3}{16}ee_i^2\begin{matrix}\cos\\ \sin\end{matrix}(2nt-3n_it-\varpi+2\varpi_i)$$

$$+\frac{9}{64}e^2e_i^2\begin{matrix}\cos\\ \sin\end{matrix}(3nt-3n_it-2\varpi+2\varpi_i)+\frac{3}{64}e^2e_i^2\begin{matrix}\cos\\ \sin\end{matrix}(nt+3n_it-2\varpi-2\varpi_i)$$

$$+\frac{e^3}{8}\begin{matrix}\cos\\ \sin\end{matrix}(nt-4n_it+3\varpi_i)-\frac{ee_i^3}{2}\begin{matrix}\cos\\ \sin\end{matrix}(4n_it-\varpi-3\varpi_i)$$

$$+\frac{ee_i^3}{6}\begin{matrix}\cos\\ \sin\end{matrix}(2nt-4n_it-\varpi+3\varpi_i)+\frac{125}{384}e_i^4\begin{matrix}\cos\\ \sin\end{matrix}(nt-5n_it+4\varpi_i)$$

$$+\frac{e_i^2}{8}\left(1+\frac{e^2}{3}e_i^2\right)\left(1-\frac{e^2}{2}\right)\begin{matrix}\cos\\ \sin\end{matrix}(nt+n_it-2\varpi_i)-\frac{3}{16}ee_i^2\begin{matrix}\cos\\ \sin\end{matrix}(n_it+\varpi-2\varpi_i)$$

$$+\frac{ee_i^2}{16}\begin{matrix}\cos\\ \sin\end{matrix}(2nt+n_it-\varpi-2\varpi_i)+\frac{3}{64}e^2e_i^2\begin{matrix}\cos\\ \sin\end{matrix}(3nt+n_it-2\varpi-2\varpi_i)$$

$$+\frac{e^2e_i^3}{64}\begin{matrix}\cos\\ \sin\end{matrix}(nt-n_it-2\varpi+2\varpi_i)+\frac{e_i^3}{24}\begin{matrix}\cos\\ \sin\end{matrix}(nt+2n_it-3\varpi_i)$$

$$-\frac{ee_i^3}{16}\begin{matrix}\cos\\ \sin\end{matrix}(2n_it+\varpi-3\varpi_i)+\frac{ee_i^3}{48}\begin{matrix}\cos\\ \sin\end{matrix}(2nt+2n_it-\varpi-3\varpi_i)$$

$$+\frac{3}{128}e_i^4\begin{matrix}\cos\\ \sin\end{matrix}(nt+3n_it-4\varpi_i)+\&c.$$

$$rr_i\begin{matrix}\cos\\ \sin\end{matrix}\{\lambda+\lambda_i\}=aa_i\left\{\left(1-\frac{e^2+e_i^2}{2}\right)\begin{matrix}\cos\\ \sin\end{matrix}(nt+n_it)\right.$$

$$-\frac{3}{2}e\begin{matrix}\cos\\ \sin\end{matrix}(n_it+\varpi)+\frac{e}{2}\begin{matrix}\cos\\ \sin\end{matrix}(2nt+n_it-\varpi)$$

$$+\frac{3}{8}e^2\begin{matrix}\cos\\ \sin\end{matrix}(3nt+n_it-2\varpi)+\frac{e^2}{8}\begin{matrix}\cos\\ \sin\end{matrix}(nt-n_it-2\varpi)$$

$$-\frac{3}{2}e_i\begin{matrix}\cos\\ \sin\end{matrix}(nt+\varpi_i)+\frac{9}{4}ee_i\begin{matrix}\cos\\ \sin\end{matrix}(\varpi+\varpi_i)-\frac{3}{4}ee_i\begin{matrix}\cos\\ \sin\end{matrix}(2nt-\varpi+\varpi_i)$$

$$+\frac{e_{,}}{2}\begin{matrix}\cos\\ \sin\end{matrix}(nt+2n_{,}t-\varpi_{,})-\frac{3}{4}ee_{,}\begin{matrix}\cos\\ \sin\end{matrix}(2n_{,}t+\varpi-\varpi_{,})+\frac{ee_{,}}{4}\begin{matrix}\cos\\ \sin\end{matrix}(2nt+2nt-\varpi-\varpi_{,})+\&c.$$

$$rr_{,}\begin{matrix}\cos\\ \sin\end{matrix}\{\lambda+\lambda_{,}-2\nu\}=aa_{,}\left\{\left(1-\frac{e^{2}+e_{,}^{2}}{2}\right)\begin{matrix}\cos\\ \sin\end{matrix}(nt+n_{,}t-2\nu)\right.$$

$$-\frac{3}{2}e\begin{matrix}\cos\\ \sin\end{matrix}(n_{,}t+\varpi-2\nu)+\frac{e}{2}\begin{matrix}\cos\\ \sin\end{matrix}(2nt+n_{,}t-\varpi-2\nu)$$

$$+\frac{3}{8}e^{2}\begin{matrix}\cos\\ \sin\end{matrix}(3nt+n_{,}t-2\varpi-2\nu)\pm\frac{e^{2}}{8}\begin{matrix}\cos\\ \sin\end{matrix}(nt-n_{,}t-2\varpi+2\nu)$$

$$-\frac{3}{2}e_{,}\begin{matrix}\cos\\ \sin\end{matrix}(nt+\varpi_{,}-2\nu)+\frac{9}{4}ee_{,}\begin{matrix}\cos\\ \sin\end{matrix}(\varpi+\varpi_{,}-2\nu)$$

$$-\frac{3}{4}ee_{,}\begin{matrix}\cos\\ \sin\end{matrix}(2nt-\varpi+\varpi_{,}-2\nu)+\frac{e_{,}}{2}\begin{matrix}\cos\\ \sin\end{matrix}(nt+2n_{,}t-\varpi_{,}-2\nu)$$

$$-\frac{3}{4}ee_{,}\begin{matrix}\cos\\ \sin\end{matrix}(2n_{,}t+\varpi-\varpi_{,}-2\nu)+\frac{ee_{,}}{4}\begin{matrix}\cos\\ \sin\end{matrix}(2nt+2n_{,}t-\varpi-\varpi_{,}-2\nu)+\&c.$$

$$r=a\left\{1+\frac{e^{2}}{2}-e\left(1-\frac{3}{8}e^{2}\right)\cos(nt-\varpi)-\frac{e^{2}}{2}\left(1-\frac{2e^{2}}{3}\right)\cos(2nt-2\varpi)\right.$$

$$-\frac{3}{8}e^{3}\cos(3nt-3\varpi)-\frac{e^{4}}{3}\cos(4nt-4\varpi)+\&c.$$

$$r^{2}=a^{2}\left\{1+\frac{3e^{2}}{2}-2e\left(1-\frac{e^{2}}{8}\right)\cos(nt-\varpi)-\frac{e^{2}}{2}\left(1-\frac{e^{2}}{3}\right)\cos(2nt-2\varpi)\right.$$

$$-\frac{e^{3}}{4}\cos(3nt-3\varpi)-\frac{e^{4}}{6}\cos(4nt-4\varpi)+\&c.$$

$$r^{-3}=a^{-3}\left\{1+\frac{3}{2}e^{2}\left(1+\frac{e^{2}}{2}\right)+3e\left(1+\frac{9}{8}e^{2}\right)\cos(nt-\varpi)\right.$$

$$+\frac{9}{2}e^{2}\left(1-\frac{e^{2}}{18}\right)\cos 2nt-2\varpi+\frac{53}{8}e^{3}\cos(3nt-3\varpi)$$

$$+\frac{31}{4}e^{4}\cos(4nt-4\varpi)+\&c.$$

If the coefficient of $\cos n\theta$ in the development of $\left\{1-\frac{2a}{a_{,}}\cos\theta+\frac{a^{2}}{a_{,}^{2}}\right\}^{-m}$ according to cosines of arcs mulitples of θ is called $b_{2m,n}$

$$b_{1,0}=1+\left(\frac{1}{2}\right)^{2}\frac{a^{2}}{a_{,}^{2}}+\left(\frac{1.3}{2.4}\right)^{2}\frac{a^{4}}{a_{,}^{4}}+\left(\frac{1.3.5}{2.4.6}\right)^{2}\frac{a^{6}}{a_{,}^{6}}+\&c.$$

$$b_{1,1} = \frac{a}{a_1} + \frac{1.3}{2.4}\frac{a^3}{a_1^3} + \frac{1.3.3.5}{2.4.4.6}\frac{a^5}{a_1^5} + \&c.$$

$$b_{1,2} = \frac{3}{4}\frac{a^2}{a_1^2} + \frac{1.3.5}{2.4.6}\frac{a^4}{a_1^4} + \frac{1.3.3.5.7}{2.4.4.6.8}\frac{a^6}{a_1^6} + \&c.$$

$$b_{1,3} = \frac{3.5}{4.6}\frac{a^3}{a_1^3} + \frac{1.3.5.7}{2.4.6.8}\frac{a^5}{a_1^5} + \frac{1.3.3.5.7.9}{2.4.4.6.8.10}\frac{a^7}{a_1^7} + \&c.$$

$$b_{3,0} = 1 + \left(\frac{3}{2}\right)^2\frac{a^2}{a_1^2} + \left(\frac{3.5}{2.4}\right)^2\frac{a^4}{a_1^4} + \left(\frac{3.5.7}{2.4.6}\right)^2\frac{a^6}{a_1^6} + \&c.$$

$$b_{3,1} = \frac{3a}{a_1} + \frac{3.3.5}{2.4}\frac{a^3}{a_1^3} + \frac{3.3.3.5.7}{2.4.4.6}\frac{a^5}{a_1^5} + \&c.$$

$$b_{3,2} = \frac{3.5}{4}\frac{a^2}{a_1^2} + \frac{3.3.5.7}{2.4.6}\frac{a^4}{a_1^4} + \frac{3.5.3.5.7}{2.4.4.6.8}\frac{a^6}{a_1^6} + \&c.$$

$$b_{3,3} = \frac{3.5.7}{4.6}\frac{a^3}{a_1^3} + \frac{3.3.5.7.9}{2.4.6.8}\frac{a^5}{a_1^5} + \frac{3.5.3.5.7.9.11}{2.4.4.6.8.10}\frac{a^7}{a_1^7} + \&c.$$

$$b_{5,0} = 1 + \left(\frac{5}{2}\right)^2\frac{a^2}{a_1^2} + \left(\frac{5.7}{2.4}\right)^2\frac{a^4}{a_1^4} + \left(\frac{5.7.9}{2.4.6}\right)^2\frac{a^6}{a_1^6} + \&c.$$

$$b_{5,1} = 5\frac{a}{a_1} + \frac{5.5.7}{2.4}\frac{a^3}{a_1^3} + \frac{5.7.5.7.9}{2.4.4.6}\frac{a^5}{a_1^5} + \&c.$$

$$b_{5,2} = \frac{5.7}{4}\frac{a^2}{a_1^2} + \frac{5.5.7.9}{2.4.6}\frac{a^4}{a_1^4} + \frac{5.7.5.7.9.11}{2.4.4.6.8}\frac{a^6}{a_1^6} + \&c.$$

$$b_{5,3} = \frac{5.7.9}{4.6}\frac{a^3}{a_1^3} + \frac{5.5.7.9.11}{2.4.6.8}\frac{a^5}{a_1^5} + \frac{5.7.5.7.9.11.13}{2.4.4.6.8.10}\frac{a^7}{a_1^7}$$

$$b_{5,1} = 3\frac{a}{a_1}\left\{b_{5,0} - \frac{1}{2}b_{5,2}\right\} \qquad b_{1,1} = \frac{a}{a_1}\left\{b_{3,0} - \frac{1}{2}b_{3,2}\right\}$$

$$2\,b_{5,2} = \frac{3}{2}\frac{a}{a_1}\left\{b_{5,1} - b_{5,3}\right\} \qquad 2\,b_{1,2} = \frac{1}{2}\frac{a}{a_1}\left\{b_{3,1} - b_{3,3}\right\}$$

$$3\,b_{5,3} = \frac{3}{2}\frac{a}{a_1}\left\{b_{5,2} - b_{5,4}\right\} \qquad 3\,b_{1,3} = \frac{1}{2}\frac{a}{a_1}\left\{b_{3,2} - b_{3,4}\right\}$$

$$R = m_r\left\{\frac{r r_1' \cos(\lambda - \lambda_r')}{r_1^3} - \frac{1}{\{r^2 - 2rr_1'\cos(\lambda - \lambda_1') + r_1^2\}^{\frac{1}{2}}}\right\}$$

and since

$$rr_r'\cos(\lambda - \lambda_r') = rr_r\left\{\cos^2\frac{i_1}{2}\cos(\lambda - \lambda_r) + \sin^2\frac{i_1}{2}\cos(\lambda + \lambda_r - 2\nu_1)\right\}, \text{ p. 330;}$$

$$R = m_1 \left\{ \frac{r r_1 \left\{ \cos^2 \frac{i_1}{2} \cos(\lambda - \lambda_1) + \sin^2 \frac{i_1}{2} \cos(\lambda + \lambda_1 - 2\nu_1) \right\}}{r_1^3} - \frac{1}{\left\{ r^2 - 2 r r_1 \left\{ \cos^2 \frac{i_1}{2} \cos(\lambda - \lambda_1) + \sin^2 \frac{i_1}{2} \cos(\lambda + \lambda_1 - 2\nu_1) \right\} + r_1^2 \right\}^{\frac{1}{2}}} \right\}$$

Neglecting the cubes and higher powers of e

$$R = m_1 \frac{a}{a_1^3} \left\{ \left(1 - \sin^2 \frac{i_1}{2} - \frac{e^2 + e_1^2}{2} \right) \cos(nt - n_1 t) - \frac{3}{2} e \cos(n_1 t - \varpi) - \frac{3}{2} e_1 \cos(nt - \varpi_1) \right.$$

$$+ \frac{e}{2} \cos(2nt - n_1 t - \varpi) + \frac{e_1}{2} \cos(nt - 2n_1 t + \varpi_1) + \frac{3}{8} e^2 \cos(3nt - n_1 t - 2\varpi)$$

$$+ \frac{3}{8} e_1^2 \cos(nt - 3n_1 t + 2\varpi_1) + \frac{e^2}{8} \cos(nt + n_1 t - 2\varpi) + \frac{e_1^2}{8} \cos(nt + n_1 t - 2\varpi_1)$$

$$+ \frac{9}{4} e e_1 \cos(\varpi - \varpi_1) - \frac{3}{4} e e_1 \cos(2nt - \varpi - \varpi_1) - \frac{3}{4} e e_1 \cos(2n_1 t - \varpi - \varpi_1)$$

$$+ \frac{e e_1}{4} \cos(2nt - 2n_1 t - \varpi + \varpi_1) + \sin^2 \frac{i}{2} \cos(nt + n_1 t - 2\nu) \Big\} \left\{ 1 + \frac{3}{2} e^2 \right.$$

$$\left. + 3 e_1 \cos(n_1 t - \varpi_1) + \frac{9}{2} e_1^2 \cos(2n_1 t - 2\varpi_1) + \&c. \right\}$$

$$- m_1 \left\{ a^2 \left\{ 1 + \frac{3}{2} e^2 - 2 e \cos(nt - \varpi) - \frac{e^2}{2} \cos(2nt - 2\varpi) \right\} \right.$$

$$- 2 a a_1 \left\{ \left(1 - \sin^2 \frac{i_1}{2} - \frac{e^2 + e_1^2}{2} \right) \cos(nt - n_1 t) - \frac{3}{2} e \cos(n_1 t - \varpi) \right.$$

$$- \frac{3}{2} e_1 \cos(nt - \varpi_1) + \frac{e}{2} \cos(2nt - n_1 t - \varpi) + \frac{e_1}{2} \cos(nt - 2n_1 t + \varpi_1)$$

$$+ \frac{3}{8} e^2 \cos(3nt - n_1 t - 2\varpi) + \frac{3}{8} e_1^2 \cos(nt - 3n_1 t + 2\varpi_1) + \frac{e^2}{8} \cos(nt + n_1 t - 2\varpi)$$

$$+ \frac{e_1^2}{8} \cos(nt + n_1 t - 2\varpi_1) + \frac{9}{4} e e_1 \cos(\varpi - \varpi_1) - \frac{3}{4} e e_1 \cos(2nt - \varpi - \varpi_1)$$

$$- \frac{3}{4} e e_1 \cos(2n_1 t - \varpi - \varpi_1) + \frac{e e_1}{4} \cos(2nt - 2n_1 t - \varpi + \varpi_1)$$

$$+ \sin^2 \frac{i}{2} \cos(nt + n_1 t - 2\nu) \Big\} + a_1^2 \left\{ 1 + \frac{3}{2} e_1^2 \right.$$

$$\left. \left. - 2 e_1 \cos(n_1 t - \varpi_1) - \frac{e_1^2}{2} \cos(2n_1 t - 2\varpi_1) + \&c. \right\} \right\}^{-\frac{1}{2}}$$

$$= \text{terms independent of } b$$

$$- \frac{m_1}{a_1}\left\{ b_{1,0} + b_{1,1}\cos(nt - n_1 t) + b_{1,2}\cos(2nt - 2n_1 t) + \&c. \right\}$$

$$+ \frac{m_1}{2a_1^3}\left\{ b_{3,0} + b_{3,1}\cos(nt - n_1 t) + b_{3,2}\cos(2nt - 2n_1 t)\ \&c. \right\} \left\{ a^2 \left\{ \frac{3}{2} e^2 \right.\right.$$

$$- 2e\cos(nt - \varpi) - \frac{e^2}{2}\cos(2nt - 2\varpi) \Big\} - 2aa_1 \left\{ \left(-\sin^2\tfrac{i}{2} - \frac{e^2 + e_1^2}{2} \right)\cos(nt - n_1 t) \right.$$

$$- \frac{3}{2} e\cos(n_1 t - \varpi) - \frac{3}{2} e_1 \cos(nt - \varpi_1) + \frac{e}{2}\cos(2nt - n_1 t - \varpi)$$

$$+ \frac{e_1}{2}\cos(nt - 2n_1 t + \varpi_1) + \frac{3}{8} e^2 \cos(3nt - n_1 t - 2\varpi) + \frac{3}{8} e_1^2 \cos(nt - 3n_1 t + 2\varpi_1)$$

$$+ \frac{e^2}{8}\cos(nt + n_1 t - 2\varpi) + \frac{e_1^2}{8}\cos(nt + n_1 t - 2\varpi_1) + \frac{9}{4} ee_1 \cos(\varpi - \varpi_1)$$

$$- \frac{3}{4} e e_1 \cos(2nt - \varpi - \varpi_1) - \frac{3}{4} ee_1 \cos(2n_1 t - \varpi - \varpi_1) + \frac{e e_1}{4}\cos(2nt - 2n_1 t - \varpi + \varpi_1)$$

$$\left. + \sin^2\tfrac{i}{2}\cos(nt + n_1 t - 2\nu_1) \right\} + a_1^2 \left\{ \frac{3}{2} e_1^2 - 2e_1 \cos(n_1 t - \varpi_1) - \frac{e_1^2}{2}\cos(2n_1 t - 2\varpi_1) \&c. \right\} \Big\}$$

$$- \frac{1.3\,m_1}{2.4\,a_1^3}\left\{ b_{5,0} + b_{5,1}\cos(nt - n_1 t) + b_{5,2}\cos(2nt - 2n_1 t) + \&c. \right\} \left\{ 2a^2 e\cos(nt - \varpi) \right.$$

$$- 3aa_1 e\cos(n_1 t - \varpi) - 3aa_1 e_1 \cos(nt - \varpi_1) + aa_1 e\cos(2nt - n_1 t - \varpi)$$

$$\left. + a\,a_1\,e_1\cos(nt - 2n_1 t + \varpi_1) + 2\,a_1^2\,e_1\cos(n_1 t - \varpi_1) \right\}^2 + \&c.$$

$$\left\{ 2\,a^2\,e\cos(nt - \varpi) - 3\,a\,a_1\,e\cos(n_1 t - \varpi) - 3\,a\,a_1\,e_1\cos(nt - \varpi_1) \right.$$

$$\left. + aa_1 e\cos(2nt - n_1 t - \varpi) + aa_1 e_1 \cos(nt - 2n_1 t + \varpi_1) + 2\,a_1^2\,e_1\cos(n_1 t - \varpi_1) \right\}^2$$

$$= 2\,a^4\,e^2 + 5\,a^2\,a_1^2\,(e^2 + e_1^2) + 2\,a_1^4\,e_1^2 + a^2(2\,a^2 - 3\,a_1^2)e^2\cos(2nt - 2\varpi)$$

$$+ \frac{9}{2} a^2 a_1^2 e^2 \cos(2n_1 t - 2\varpi) + \frac{9}{2} a^2 a_1^2 e_1^2 \cos(2nt - 2\varpi_1) + \frac{a^2 a_1^2}{2} e^2 \cos(4nt - 2n_1 t - 2\varpi)$$

$$+ \frac{a^2 a_1^2}{2} e_1^2 \cos(2nt - 4n_1 t + 2\varpi_1) + a_1^2 (2\,a_1^2 - 3\,a^2)\,e_1^2 \cos(2n_1 t - 2\varpi_1)$$

$$- 4\,(a^2 e^2 + a_1^2 e_1^2) aa_1 \cos(nt - n_1 t) - 6\,a^3 a_1 e^2 \cos(nt + n_1 t - 2\varpi)$$

$$- 6\, a a_{,}^{3} e_{,}^{2} \cos(nt + n_{,}t - 2\varpi_{,}) - 6\,(a^2 + a_{,}^{2})\, a a_{,} e e_{,} \cos(\varpi - \varpi_{,})$$

$$- 2\,(3a^2 - a_{,}^{2})\, a a_{,} e e_{,} \cos(2nt - \varpi - \varpi_{,}) - 2\,(3a_{,}^{2} - a^2)\, a a_{,} e e_{,} \cos(2n_{,}t - \varpi - \varpi_{,})$$

$$+ 2\, a^3 a_{,} e^2 \cos(3nt - n_{,}t - 2\varpi) + 2\, a a_{,}^{3} e_{,}^{2} \cos(nt - 3n_{,}t + 2\varpi_{,})$$

$$+ 2\,(a^2 + a_{,}^{2})\, a a_{,} e e_{,} \cos(2nt - 2n_{,}t - \varpi + \varpi_{,}) + 9\, a^2 a_{,}^{2} e e_{,} \cos(nt - n_{,}t + \varpi - \varpi_{,})$$

$$+ 14\, a^2 a_{,}^{2} e e_{,} \cos(nt + n_{,}t - \varpi - \varpi_{,}) - 3\, a^2 a_{,}^{2} (e^2 + e_{,}^{2}) \cos(2nt - 2n_{,}t)$$

$$- 3\, a^2 a_{,}^{2} e e_{,} \cos(nt - 3n_{,}t + \varpi + \varpi_{,}) - 3\, a^2 a_{,}^{2} e e_{,} \cos(3nt - n_{,}t - \varpi - \varpi_{,})$$

$$- 2\, a^2 a_{,}^{2} e e_{,} \cos(nt - n_{,}t - \varpi + \varpi_{,}) + a^3 a_{,}^{2} e e_{,} \cos(3nt - 3n_{,}t - \varpi + \varpi_{,})$$

Replacing i which accompanies nt,

$$R = m_{,} \Big\{ -\frac{b_{1,0}}{a_{,}} + 3\, \frac{(a^2 e^2 + a_{,}^{2} e_{,}^{2})}{2.2\, a_{,}^{3}} b_{3,0} + \frac{a\, a_{,}}{2\, a_{,}^{2}} \Big(\sin^2 \frac{i_{,}}{2} + \frac{e^2 + e_{,}^{2}}{2} \Big) b_{3,1}$$

$$- \frac{1.3}{2.4.a_{,}^{3}} \Big(2a^4 e^2 + 5a^2 a_{,}^{2} (e^2 + e_{,}^{2}) + 2a_{,}^{4} e_{,}^{2} \Big) b_{5,0} + \frac{1.3.2}{2.4\, a_{,}^{3}} (a^2 e^2 + a_{,}^{2} e_{,}^{2})\, a a_{,} b_{5,1}$$

$$+ \frac{1.3.3}{2.4.2} \frac{a^2 a_{,}^{2}}{a_{,}^{5}} (e^2 + e_{,}^{2}) b_{5,2} \qquad [0^*]$$

$$+ m_{,} \Big\{ \frac{a}{a_{,}^{2}} \Big\{ \cos^2 \frac{i_{,}}{2} - \frac{e^2 + e_{,}^{2}}{2} \Big\} - \frac{b_{1,1}}{a_{,}} + \frac{a\, a_{,}}{a_{,}^{2}} \Big\{ \sin^2 \frac{i_{,}}{2} + \frac{e^2 + e_{,}^{2}}{2} \Big\} \Big\{ b_{3,0} + \tfrac{1}{2}\, b_{3,2} \Big\}$$

$$+ \frac{3\,(a^2 e^2 + a_{,}^{2} e_{,}^{2})}{2.2\, a_{,}^{3}} b_{3,1} + \frac{1.3.4}{2.4\, a_{,}^{5}} (a^2 e^2 + a_{,}^{2} e_{,}^{2})\, a a_{,} b_{5,0}$$

$$- \frac{1.3}{2.4\, a_{,}^{5}} \Big\{ 2a^4 e^2 + \frac{7}{2} a^2 a_{,}^{2} (e^2 + e_{,}) + 2a_{,}^{4} e_{,}^{2} \Big\} b_{5,1} + \frac{1.3.2}{2.4\, a_{,}^{5}} (a^2 e^2 + a_{,}^{2} e_{,}^{2})\, a a_{,} b_{5,2}$$

$$+ \frac{1.3.3}{2.4.2} \frac{a^2 a_{,}^{2}}{a_{,}^{5}} (e^2 + e_{,}^{2}) b_{5,3} \Big\} \cos(nt - n_{,}t + \varepsilon - \varepsilon_{,}) \qquad [1^*]$$

$$+ m_{,} \Big\{ -\frac{b_{1,2}}{a_{,}} + \frac{a\, a_{,}}{2\, a_{,}^{2}} \Big\{ \sin^2 \frac{i_{,}}{2} + \frac{e^2 + e_{,}^{2}}{2} \Big\} \Big\{ b_{3,1} + b_{3,3} \Big\} + \frac{3\,(a^2 e^2 + a_{,}^{2} e_{,}^{2})}{2.2 a_{,}^{3}} b_{3,2}$$

$$+ \frac{1.3.3}{2.4} \frac{a^2 a_{,}^{2} (e^2 + e_{,}^{2})}{a_{,}^{5}} b_{5,0} + \frac{1.3.2}{2.4} \frac{(a^2 e^2 + a_{,}^{2} e_{,}^{2})}{a_{,}^{5}} a a_{,} b_{5,1}$$

* The numbers at the side serve to distinguish the arguments.

$$-\frac{1.3}{2.4\,a_i^3}\left\{2a^4e^2+5a^2a_i^2(e^2+e_i^2)+2a_i^4e_i^2\right\}b_{3,2}$$

$$+\frac{1.3.2}{2.4\,a_i^3}(a^2e^2+a_i^2e_i^2)b_{3,3}\Big\}\cos(2nt-2n_it+2\varepsilon-2\varepsilon_i) \qquad [2]$$

$$+m_i\Big\{-\frac{b_{1,3}}{a_i}+\frac{a\,a_i}{2\,a_i^3}\Big\{\sin^2\frac{i_i}{2}+\frac{e^2+e_i^2}{2}\Big\}b_{3,2}+\frac{3(a^2e^2+a_i^2e_i^2)}{2.2\,a_i^3}b_{3,3}$$

$$+\frac{1.3.3}{2.4.2}\frac{a^3a_i^3(e^3+e_i^3)}{a_i^5}b_{3,1}+\frac{1.3.2}{2.4}\frac{(a^2e^2+a_i^2e_i^2)}{4a_i^5}aa_ib_{3,2}$$

$$-\frac{1.3}{2.4}\left\{2a^4e^2+5a^2a_i^2(e^2+e_i^2)+2a_i^4e_i^2\right\}b_{3,3}\Big\}\cos(3nt-3n_it+3\varepsilon-3\varepsilon_i) \qquad [3]$$

$$+m_i\Big\{-\frac{b_{1,4}}{a_i}+\frac{a\,a_i}{2\,a_i^3}\Big\{\sin^2\frac{i_i}{2}+\frac{e^2+e_i^2}{2}\Big\}b_{3,3}+\frac{1.3.3}{2.4.2}\frac{a^3a_i^3(e^3+e_i^3)}{a_i^5}b_{3,2}$$

$$+\frac{1.3.2}{2.4}\frac{(a^2e^2+a_i^2e_i^2)}{a_i^5}aa_ib_{3,3}\Big\}\cos(4nt-4n_it+4\varepsilon-4\varepsilon_i) \qquad [4]$$

$$+m_i\Big\{-\frac{b_{1,5}}{a_i}+\frac{1.3.3}{2.4.2}\frac{a^3a_i^3(e^3+e_i^3)}{a_i^5}b_{3,3}\Big\}\cos(5nt-5n_it+5\varepsilon-5\varepsilon_i) \qquad [5]$$

$$+m_i\Big\{-\frac{3\,a}{2\,a_i^3}+\frac{3\,a\,a'}{2\,a_i^3}b_{3,0}-\frac{a^2}{2\,a_i^3}b_{3,1}-\frac{a\,a_i}{2.2\,a_i^3}b_{3,2}\Big\}e\cos(n_it+\varepsilon_i-\varpi) \qquad [6]$$

$$+m_i\Big\{-\frac{a^2}{a_i^3}b_{3,0}+\frac{a\,a_i}{2\,a_i^3}b_{3,1}\Big\}e\cos(nt+\varepsilon-\varpi) \qquad [7]$$

$$+m_i\Big\{\frac{1}{2}\frac{a}{a_i^3}-\frac{a\,a_i}{2\,a_i^3}b_{3,0}-\frac{a^2}{2\,a_i^3}b_{3,1}+\frac{3}{2.2}\frac{a\,a_i}{a_i^3}b_{3,2}\Big\}e\cos(2nt-n_it+2\varepsilon-\varepsilon_i-\varpi) \qquad [8]$$

$$+m_i\Big\{-\frac{a\,a_i}{2.2\,a_i^3}b_{3,1}-\frac{a^2}{2\,a_i^3}b_{3,2}+\frac{3}{2.2}\frac{a\,a_i}{a_i^3}b_{3,3}\Big\}e\cos(3nt-2n_it+3\varepsilon-2\varepsilon_i-\varpi) \qquad [9]$$

$$+m_i\Big\{-\frac{a\,a_i}{2.2\,a_i^3}b_{3,2}-\frac{a^2}{2\,a_i^3}b_{3,3}\Big\}e\cos(4nt-3n_it+4\varepsilon-3\varepsilon_i-\varpi) \qquad [10]$$

$$-m_i\frac{a\,a_i}{2.2\,a_i^3}b_{3,3}\,e\cos(5nt-4n_it+5\varepsilon-4\varepsilon_i-\varpi) \qquad [11]$$

$$+m_i\Big\{\frac{3}{2.2}\frac{a\,a_i}{a_i^3}b_{3,1}-\frac{a^2}{2\,a_i^3}b_{3,2}-\frac{a\,a_i}{2.2\,a_i^3}b_{3,3}\Big\}e\cos(nt-2n_it+\varepsilon-2\varepsilon_i+\varpi) \qquad [12]$$

$$+m_i\Big\{\frac{3}{2.2}\frac{a\,a_i}{a_i^3}b_{3,2}-\frac{a^2}{2\,a_i^3}b_{3,3}\Big\}e\cos(2nt-3n_it+2\varepsilon-3\varepsilon_i+\varpi) \qquad [13]$$

$$+m_i\frac{3\,a\,a_i}{2.2\,a_i^3}b_{3,3}\,e\cos(3nt-4n_it+3\varepsilon-4\varepsilon_i+\varpi) \qquad [14]$$

$$+ m_{,} \left\{ - \frac{a_{,}^{2}}{a_{,}^{2}} b_{2,0} - \frac{a a_{,}}{2 a_{,}^{2}} b_{2,1} \right\} e_{,} \cos(n_{,} t + \varepsilon_{,} - \varpi_{,}) \quad [15]$$

$$+ m_{,} \left\{ \frac{3 a a_{,}}{2 . 2 a_{,}^{2}} b_{2,1} - \frac{a_{,}^{2}}{2 a_{,}^{2}} b_{2,2} - \frac{a a_{,}}{2 . 2 a_{,}^{2}} b_{2,3} \right\} e_{,} \cos(2nt - n_{,} t + 2\varepsilon - \varepsilon_{,} - \varpi_{,}) \quad [17]$$

$$+ m_{,} \left\{ \frac{3 a a_{,}}{2 . 2 a_{,}^{2}} b_{2,2} - \frac{a_{,}^{2}}{2 a_{,}^{2}} b_{2,3} \right\} e_{,} \cos(3nt - 2n_{,} t + 3\varepsilon - 2\varepsilon_{,} - \varpi_{,}) \quad [18]$$

$$+ m_{,} \frac{3 a a_{,}}{2 . 2 a_{,}^{2}} b_{2,3} e_{,} \cos(4nt - 3n_{,} t + 4\varepsilon - 3\varepsilon_{,} - \varpi_{,}) \quad [19]$$

$$+ m_{,} \left\{ \frac{2 a}{a_{,}^{2}} - \frac{a a_{,}}{2 a_{,}^{2}} b_{2,0} - \frac{a_{,}^{2}}{2 a_{,}^{2}} b_{2,1} + \frac{3}{2 . 2} \frac{a a_{,}}{a_{,}^{2}} b_{2,2} \right\} e_{,} \cos(nt - 2n_{,} t + \varepsilon - 2\varepsilon_{,} + \varpi_{,}) \quad [20]$$

$$+ m_{,} \left\{ - \frac{a a_{,}}{2 . 2 a_{,}^{2}} b_{2,1} - \frac{a_{,}^{2}}{2 a_{,}^{2}} b_{2,2} + \frac{3}{2 . 2} \frac{a a_{,}}{a_{,}^{2}} b_{2,3} \right\} e_{,} \cos(2nt - 3n_{,} t + 2\varepsilon - 3\varepsilon_{,} + \varpi_{,}) \quad [21]$$

$$+ m_{,} \left\{ - \frac{a a_{,}}{2 . 2 a_{,}^{2}} b_{2,2} - \frac{a_{,}^{2}}{2 a_{,}^{2}} b_{2,3} \right\} e_{,} \cos(3nt - 4n_{,} t + 3\varepsilon - 4\varepsilon_{,} + \varpi_{,}) \quad [22]$$

$$- m_{,} \frac{a a_{,}}{2 . 2 a_{,}^{2}} b_{2,3} e_{,} \cos(4nt - 5n_{,} t + 4\varepsilon - 5\varepsilon_{,} + \varpi_{,}) \quad [23]$$

$$+ m_{,} \left\{ - \frac{a a_{,}}{2 . 8 a_{,}^{2}} b_{2,1} - \frac{a^{2}}{2 . 4 a_{,}^{2}} b_{2,2} - \frac{3}{2 . 8} \frac{a a_{,}}{a_{,}^{2}} b_{2,3} - \frac{1 . 3 . 9}{2 . 4 . 2} \frac{a^{2} a_{,}^{2}}{a_{,}^{4}} b_{2,0} + \frac{1 . 3 . 3}{2 . 4} \frac{a^{2} a_{,}}{a_{,}^{3}} b_{2,1} \right.$$
$$\left. - \frac{1 . 3}{2 . 4 . 2} \frac{(2 a^{2} - 3 a_{,}^{2})}{a_{,}^{2}} a^{2} b_{2,2} - \frac{1 . 3}{2 . 4} \frac{a^{2} a_{,}}{a_{,}^{3}} b_{2,3} \right\} e^{2} \cos(2n_{,} t + 2\varepsilon_{,} - 2\varpi) \quad [24]$$

$$+ m_{,} \left\{ \frac{a}{8 a_{,}^{2}} - \frac{a a_{,}}{2 . 4 a_{,}^{3}} b_{2,0} - \frac{a^{2}}{2 . 4 a_{,}^{3}} b_{2,1} - \frac{3 a a_{,}}{2 . 8 a_{,}^{3}} b_{2,2} + \frac{1 . 3 . 6}{2 . 4} \frac{a^{2} a_{,}}{a_{,}^{5}} b_{2,0} \right.$$
$$- \frac{1 . 3 a^{2}}{2 . 4 . 4} \frac{(4 a^{2} + 3 a_{,}^{2})}{a_{,}^{5}} b_{2,1} - \frac{1 . 3}{2 . 4} \frac{a^{2} a_{,}}{a_{,}^{3}} b_{2,2}$$
$$\left. \frac{1 . 3}{2 . 4 . 4} \frac{a^{2} a_{,}^{2}}{a_{,}^{5}} b_{2,3} \right\} e^{2} \cos(nt + n_{,} t + \varepsilon + \varepsilon_{,} - 2\varpi) \quad [25]$$

$$+ m_{,} \left\{ - \frac{a^{2}}{2 . 2 a_{,}^{3}} b_{2,0} - \frac{a a_{,}}{2 . 2 a_{,}^{3}} b_{2,1} - \frac{1 . 3}{2 . 4} \frac{a^{2} (2 a^{2} - 3 a_{,}^{2})}{a_{,}^{5}} b_{2,0} \right.$$
$$\left. + \frac{1 . 3 . 2}{2 . 4} \frac{a^{2} a_{,}}{a_{,}^{5}} b_{2,1} - \frac{1 . 3 . 5}{2 . 4 . 2} \frac{a^{2} a_{,}^{2}}{a_{,}^{5}} b_{2,2} \right\} e^{2} \cos(2nt + 2\varepsilon - 2\varpi) \quad [26]$$

$$+ m_{,} \left\{ \frac{3}{8} \frac{a}{a_{,}^{2}} - \frac{3}{8} \frac{a a_{,}}{a_{,}^{3}} b_{2,0} - \frac{a a_{,}}{2 . 4 a_{,}^{3}} b_{2,1} - \frac{a^{2}}{2 . 8 a_{,}^{3}} b_{2,2} - \frac{1 . 3 . 2}{2 . 4} \frac{a^{2} a_{,}}{a_{,}^{5}} b_{2,0} \right.$$
$$\left. - \frac{1 . 3 a^{2}}{2 . 4 . 4} \frac{(4 a^{2} - 5 a_{,}^{2})}{a_{,}^{5}} b_{2,1} + \frac{1 . 3 . 3}{2 . 4} \frac{a^{2} a_{,}}{a_{,}^{5}} b_{2,2} \right.$$

$$- \frac{1.3.9}{2.4.4}\frac{a^3 a_1^3}{a_1^5} b_{5,3}\Big\} e^2 \cos(3nt - n_1 t + 3\varepsilon - \varepsilon_1 - 2\varpi) \quad [27]$$

$$+ m_1\Big\{ - \frac{3\,a\,a_1}{2.8\,a_1^3} b_{3,1} - \frac{a^3}{2.4\,a_1^3} b_{3,2} - \frac{a\,a_1}{2.8\,a_1^3} b_{3,3} - \frac{1.3\,a^3 a_1^3}{2.4.2\,a_1^5} b_{5,0}$$

$$- \frac{1.3}{2.4}\frac{a^3 a_1}{a_1^5} b_{5,1} - \frac{1.3\,a^3}{2.4.2}\frac{(2a^3 - 3a_1^3)}{a_1^5} b_{5,2}$$

$$+ \frac{1.3.3}{2.4}\frac{a^3 a_1}{a_1^5} b_{5,3}\Big\} e^2 \cos(4nt - 2n_1 t + 4\varepsilon - 2\varepsilon_1 - 2\varpi) \quad [28]$$

$$+ m_1\Big\{ - \frac{3\,a\,a_1}{2.8\,a_1^3} b_{3,2} - \frac{a^3}{2.4\,a_1^3} b_{3,3} - \frac{1.3}{2.4.4}\frac{a^3 a_1^3}{a_1^5} b_{5,1} - \frac{1.3}{2.4}\frac{a^3 a_1}{a_1^5} b_{5,2}$$

$$- \frac{1.3\,a^3}{2.4.2}\frac{(2a^3 - 3a_1^3)}{a_1^5} b_{5,3}\Big\} e^2 \cos(5nt - 3n_1 t + 5\varepsilon - 3\varepsilon_1 - 2\varpi) \quad [29]$$

$$+ m_1\Big\{ - \frac{3\,a a_1}{2.8\,a_1^3} b_{3,3} - \frac{1.3}{2.4.4}\frac{a^3 a_1^3}{a_1^5} b_{5,2}$$

$$- \frac{1.3}{2.4}\frac{a^3 a_1}{a_1^5} b_{5,3}\Big\} e^2 \cos(6nt - 4n_1 t + 6\varepsilon - 4\varepsilon_1 - 2\varpi) \quad [30]$$

$$- m_1 \frac{1.3}{2.4.4}\frac{a^3 a_1^3}{a_1^5} b_{5,3}\, e^2 \cos(7nt - 5n_1 t + 7\varepsilon - 5\varepsilon_1 - 2\varpi) \quad [31]$$

$$+ m_1\Big\{ - \frac{a a_1}{2.8\,a_1^3} b_{3,3} - \frac{a^3}{2.4\,a_1^3} b_{3,3} - \frac{1.3.9}{2.4}\frac{a^3 a_1^3}{a_1^5} b_{5,1} + \frac{1.3.3}{2.4}\frac{a^3 a_1}{a_1^5} b_{5,3}$$

$$- \frac{1.3}{2.4}\frac{a^3}{2}\frac{(2a^3 - 3a_1^3)}{a_1^5} b_{5,3}\Big\} e^2 \cos(nt - 3n_1 t + \varepsilon - 3\varepsilon_1 + 2\varpi) \quad [32]$$

$$- m_1\Big\{ - \frac{a a_1}{2.8\,a_1^3} b_{3,3} - \frac{1.3.9}{2.4.4}\frac{a^3 a_1^3}{a_1^5} b_{5,3}$$

$$+ \frac{1.3.3}{2.4}\frac{a^3 a_1}{a_1^5} b_{5,3}\Big\} e^2 \cos(2nt - 4n_1 t + 2\varepsilon - 4\varepsilon_1 + 2\varpi) \quad [33]$$

$$- m_1 \frac{1.3.9}{2.4.4}\frac{a^3 a_1^3}{a_1^5} b_{5,3}\, e^2 \cos(3nt - 5n_1 t + 3\varepsilon - 5\varepsilon_1 + 2\varpi) \quad [34]$$

$$+ m_1\Big\{ - \frac{5}{2.2}\frac{a a_1}{a_1^3} b_{3,1} + \frac{1.3.2}{2.4}\frac{a^3 a_1^3}{a_1^5} b_{5,0} + \frac{1.3.2}{2.4}\frac{(a^3 + a_1^3)}{a_1^5} a a_1 b_{5,1}$$

$$- \frac{1.3.5}{2.4}\frac{a^3 a_1^3}{a_1^5} b_{5,3}\Big\} ee_1 \cos(nt - n_1 t + \varepsilon - \varepsilon_1 - \varpi + \varpi_1) \quad [35]$$

$$+ m_1\Big\{ \frac{a}{a_1^3} - \frac{a a_1}{2.2\,a_1^3} b_{3,0} - \frac{9}{2.4}\frac{a a_1}{a_1^3} b_{3,2} - \frac{1.3.2}{2.4}\frac{(a^3 + a_1^3)}{a_1^5} a a_1 b_{5,2}$$

$$+\frac{1.3}{2.4.2}\frac{a^2a_1^2}{a_1^5}b_{5,1}+\frac{1.3.3}{2.4}\frac{a^3+a_1^3}{a_1^5}aa_1b_{5,2}$$

$$-\frac{1.3.9}{2.4.2}\frac{a^2a_1^2}{a_1^5}b_{5,3}\Big\}ee_1\cos(2nt-2n_1t+2\varepsilon-2\varepsilon_1-\varpi+\varpi_1) \quad [36]$$

$$+m_1\Big\{-\frac{aa_1}{2.4a_1^3}b_{3,1}-\frac{9}{2.4}\frac{aa_1}{a_1^3}b_{3,2}-\frac{1.3}{2.4}\frac{a^2a_1^2}{a_1^5}b_{5,0}-\frac{1.3}{2.4}\frac{(a^3+a_1^3)}{a_1^5}aa_1b_{5,1}+\frac{1.3}{2.4}\frac{a^2a_1^2}{a_1^5}b_{5,2}$$

$$+\frac{1.3.3}{2.4}\frac{a^3+a_1^3}{a_1^5}aa_1b_{5,3}\Big\}ee_1\cos(3nt-3n_1t+3\varepsilon-3\varepsilon_1-\varpi+\varpi_1) \quad [37]$$

$$+m_1\Big\{-\frac{aa_1}{2.4a_1^3}b_{3,2}-\frac{1.3}{2.4.2}\frac{a^2a_1^2}{a_1^5}b_{5,1}-\frac{1.3}{2.4}\frac{(a^3+a_1^3)}{a_1^5}aa_1b_{5,2}$$

$$+\frac{1.3}{2.4}\frac{a^2a_1^2}{a_1^5}b_{5,3}\Big\}ee_1\cos(4nt-4n_1t+4\varepsilon-4\varepsilon_1-\varpi+\varpi_1) \quad [38]$$

$$+m_1\Big\{-\frac{aa_1}{2.4a_1^3}b_{3,3}-\frac{1.3}{2.4.2}\frac{a^2a_1^2}{a_1^5}b_{5,2}$$

$$-\frac{1.3}{2.4}\frac{(a^3+a_1^3)}{a_1^5}aa_1b_{5,3}\Big\}ee_1\cos(5nt-5n_1t+5\varepsilon-5\varepsilon_1-\varpi+\varpi_1) \quad [39]$$

$$-m_1\frac{1.3}{2.4}\frac{a^2a_1^2}{2a_1^5}b_{5,3}\cos(6nt-6n_1t-\varpi+\varpi_1) \quad [40]$$

$$+m_1\Big\{-\frac{9}{2.2}\frac{aa_1}{a_1^3}b_{3,0}-\frac{aa_1}{2.4a_1^3}b_{3,3}+\frac{1.3.6}{2.4}\frac{(a^3+a_1^3)}{a_1^5}aa_1b_{5,0}-\frac{1.3.7}{2.4.2}\frac{a^3a_1^3}{a_1^5}b_{5,1}$$

$$-\frac{1.3}{2.4}\frac{(a^3+a_1^3)}{a_1^5}a_1^2b_{5,2}-\frac{1.3}{2.4.2}\frac{a^3a_1^3}{a_1^5}b_{5,3}\Big\}ee_1\cos(\varpi-\varpi_1) \quad [41]$$

$$+m_1\Big\{-\frac{9}{2.4}\frac{aa_1}{a_1^3}b_{3,1}-\frac{aa_1}{2.4a_1^3}b_{3,3}-\frac{1.3.9}{2.4}\frac{a^2a_1^2}{a_1^5}b_{5,0}+\frac{1.3.3}{2.4}\frac{(a^3+a_1^3)}{a_1^5}aa_1b_{5,1}$$

$$+\frac{1.3}{2.4}\frac{a^2a_1^2}{a_1^5}b_{5,2}-\frac{1.3}{2.4}\frac{(a^3+a_1^3)}{a_1^5}aa_1b_{5,3}\Big\}ee_1\cos(nt-n_1t+\varepsilon-\varepsilon_1+\varpi-\varpi_1) \quad [42]$$

$$+m_1\Big\{-\frac{9}{2.4}\frac{aa_1}{a_1^3}b_{3,2}-\frac{1.3.9}{2.4.2}\frac{a^2a_1^2}{a_1^5}b_{5,1}+\frac{1.3.3}{2.4}\frac{(a^3+a_1^3)}{a_1^5}aa_1b_{5,2}$$

$$+\frac{1.3}{2.4}\frac{a^2a_1^2}{a_1^5}b_{5,3}\Big\}ee_1\cos(2nt-2n_1t+2\varepsilon-2\varepsilon_1+\varpi-\varpi_1) \quad [43]$$

$$+m_1\Big\{-\frac{9}{2.4}\frac{aa_1}{a_1^3}b_{3,3}-\frac{1.3.9}{2.4.2}\frac{a^2a_1^2}{a_1^5}b_{5,2}$$

$$+\frac{1.3.3}{2.4}\frac{(a^3+a_1^3)}{a_1^5}aa_1b_{5,3}\Big\}ee_1\cos(3nt-3n_1t+3\varepsilon-3\varepsilon_1+\varpi-\varpi_1) \quad [44]$$

$$- m, \frac{1.3.9}{2.4.2} \frac{a^3 a_1^3}{a_1^5} b_{5,3}\, ee, \cos(4nt - 4n't + 4\varepsilon - 4\varepsilon, + \varpi - \varpi,) \quad [45]$$

$$+ m, \left\{ - \frac{3}{2} \frac{a}{a_1^2} + \frac{3}{2.2} \frac{a a_1}{a_1^3} b_{3,0} + \frac{3}{2.4} \frac{a a_1}{a_1^3} b_{3,2} + \frac{1.3.2}{1.4} \frac{(3a_1^2 - a^2)}{a_1^5} a a_1 b_{3,0} - \frac{1.3.11}{2.4.2} \frac{a^3 a_1^3}{a_1^5} b_{5,1} \right.$$
$$\left. + \frac{1.3}{2.4} \frac{(3a_1^2 - a^2)}{a_1^5} a a_1 b_{5,2} + \frac{1.3.3}{2.4.2} \frac{a^3 a_1^3}{a_1^5} b_{5,3} \right\} ee, \cos(2n't + 2\varepsilon, - \varpi - \varpi,) \quad [46]$$

$$+ m, \left\{ \frac{3}{2.2} \frac{a a_1}{a_1^3} b_{3,1} - \frac{1.3.14}{2.4} \frac{a^3 a_1^3}{a_1^5} b_{5,0} + \frac{1.3.2}{2.4} \frac{(a^2 + a_1^2)}{a_1^5} a a_1 b_{5,1} \right.$$
$$\left. + \frac{1.3.3}{2.4} \frac{a^3 a_1^3}{a_1^5} b_{5,3} \right\} ee, \cos(nt + n't + \varepsilon + \varepsilon, - \varpi - \varpi,) \quad [47]$$

$$+ m, \left\{ \frac{3}{2.2} \frac{a a_1}{a_1^3} b_{3,0} + \frac{3}{2.4} \frac{a a_1}{a_1^3} b_{3,2} + \frac{1.3.2}{2.4} \frac{(3a^2 - a_1^2)}{a_1^5} a a_1 b_{5,0} - \frac{1.3.11}{2.4.2} \frac{a^3 a_1^3}{a_1^5} b_{5,1} \right.$$
$$\left. + \frac{1.3}{2.4} \frac{(3a_1^2 - a^2)}{a_1^5} a a_1 b_{5,2} + \frac{1.3.3}{2.4.2} \frac{a^3 a_1^3}{a_1^5} b_{5,3} \right\} ee, \cos(2nt + 2\varepsilon - \varpi - \varpi,) \quad [48]$$

$$+ m, \left\{ \frac{3}{2.4} \frac{a a_1}{a_1^3} b_{3,1} + \frac{3}{2.4} \frac{a a_1}{a_1^3} b_{3,3} + \frac{1.3.3}{2.4} \frac{a^3 a_1^3}{a_1^5} b_{5,0} \right.$$
$$+ \frac{1.3}{2.4} \frac{(3a^2 - a_1^2)}{a_1^5} a a_1 b_{5,1} - \frac{1.3.7}{2.4} \frac{a^3 a_1^3}{a_1^5} b_{5,2}$$
$$\left. + \frac{1.3}{2.4} \frac{(3a_1^2 - a^2)}{a_1^5} a a_1 b_{5,3} \right\} ee, \cos(3nt - n't + 3\varepsilon - \varepsilon, - \varpi - \varpi,) \quad [49]$$

$$+ m, \left\{ \frac{3}{2.4} \frac{a a_1}{a_1^3} b_{3,2} + \frac{1.3.3}{2.4.2} \frac{a^3 a_1^3}{a_1^5} b_{5,1} + \frac{1.3}{2.4} \frac{(3a^2 - a_1^2)}{a_1^5} a a_1 b_{5,2} \right.$$
$$\left. - \frac{1.3.7}{2.4} \frac{a^3 a_1^3}{a_1^5} b_{5,3} \right\} ee, \cos(4nt - 2n't + 4\varepsilon - 2\varepsilon, - \varpi - \varpi,) \quad [50]$$

$$+ m, \left\{ \frac{3}{2.4} \frac{a a_1}{a_1^3} b_{3,3} + \frac{1.3.3}{2.4.2} \frac{a^3 a_1^3}{a_1^5} a a_1 b_{5,2} \right.$$
$$\left. + \frac{1.3}{2.4} \frac{(3a^2 - a_1^2)}{a_1^5} a a_1 b_{5,3} \right\} ee, \cos(5nt - 3n't + 5\varepsilon - 3\varepsilon, - \varpi - \varpi,) \quad [51]$$

$$+ m, \frac{1.3.3}{2.4.2} \frac{a^3 a_1^3}{a_1^5} b_{5,3}\, ee, \cos(6nt - 4n't + 6\varepsilon - 4\varepsilon, - \varpi - \varpi,) \quad [52]$$

$$+ m, \left\{ \frac{3}{2.4} \frac{a a_1}{a_1^3} b_{3,1} + \frac{3}{2.4} \frac{a a_1}{a_1^3} b_{3,2} + \frac{1.3.3}{2.4} \frac{a^3 a_1^3}{a_1^5} b_{5,0} \right.$$
$$\left. + \frac{1.3}{2.4} \frac{(3a_1^2 - a^2)}{a_1^5} a a_1 b_{5,1} - \frac{1.3.7}{2.4} \frac{a^3 a_1^3}{a_1^5} b_{5,2} \right.$$

$$+\frac{1.3}{2.4}\frac{(3a^2-a_1^2)}{a_1^2}aa_1b_{3,3}\Big\}ee_1\cos(nt-3n_1t+\varepsilon-3\varepsilon_1+\varpi+\varpi_1) \quad [53]$$

$$+m_1\Big\{\frac{3}{2.4}\frac{aa_1}{a_1^2}b_{3,1}+\frac{3}{2.4}\frac{aa_1}{a_1^2}b_{3,2}+\frac{1.3.3}{2.4}\frac{a^2a_1^2}{a_1^4}b_{3,0}$$
$$+\frac{1.3}{2.4}\frac{(3a_1^2-a^2)}{a_1^2}aa_1b_{3,1}+\frac{1.3}{2.4}\frac{(3a^2-a_1^2)}{a_1^2}aa_1b_{3,2}$$
$$-\frac{1.3.7}{2.4}\frac{a^2a_1^2}{a_1^4}b_{3,3}\Big\}ee_1\cos(2nt-4n_1t+2\varepsilon-4\varepsilon_1+\varpi+\varpi_1) \quad [54]$$

$$+m_1\Big\{\frac{3}{2.4}\frac{aa_1}{a_1^2}b_{3,3}+\frac{1.3.3}{2.4.2}\frac{a^2a_1^2}{a_1^4}b_{3,2}$$
$$+\frac{1.3}{2.4}\frac{(3a_1^2-a^2)}{a_1^2}aa_1b_{3,3}\Big\}ee_1\cos(3nt-5n_1t+3\varepsilon-5\varepsilon_1+\varpi+\varpi_1) \quad [55]$$

$$+m_1\frac{1.3.3}{2.4.2}\frac{a^2a_1^2}{a_1^4}b_{3,3}ee_1\cos(4nt-6n_1t+4\varepsilon-6\varepsilon_1+\varpi+\varpi_1) \quad [56]$$

$$+m_1\Big\{-\frac{a_1^2}{2.2a_1^2}b_{2,0}-\frac{aa_1}{2.2a_1^2}b_{2,1}-\frac{1.3}{2.4}\frac{(2a_1^2-3a^2)}{a_1^4}a_1^2b_{2,0}$$
$$+\frac{1.3.2}{2.4}\frac{aa_1^2}{a_1^4}b_{2,1}-\frac{1.3.5}{2.4.2}\frac{a^2a_1^2}{a_1^4}b_{2,2}\Big\}e_1^2\cos(2n_1t+2\varepsilon_1-2\varpi_1) \quad [57]$$

$$+m_1\Big\{\frac{a}{8a_1^2}-\frac{aa_1}{2.4a_1^2}b_{2,0}-\frac{a_1^2}{2.4a_1^2}b_{2,1}-\frac{3aa_1}{2.8a_1^2}b_{2,2}+\frac{1.3.6}{2.4}\frac{a_1^2a}{a_1^4}b_{2,0}$$
$$-\frac{1.3}{2.4.4}\frac{(4a^2+3a_1^2)}{a_1^4}a_1^2b_{2,1}-\frac{1.3}{2.4}\frac{aa_1^2}{a_1^4}b_{2,2}$$
$$-\frac{1.3}{2.4.4}\frac{a^2a_1^2}{a_1^4}b_{2,3}\Big\}e_1^2\cos(nt+n_1t+\varepsilon+\varepsilon_1-2\varpi_1) \quad [58]$$

$$+m_1\Big\{-\frac{aa_1}{2.8a_1^2}b_{2,1}-\frac{a_1^2}{2.4a_1^2}b_{2,2}-\frac{3}{2.8}\frac{aa_1}{a_1^2}b_{2,3}-\frac{1.3.9}{2.4.2}\frac{a^2a_1^2}{a_1^4}b_{2,0}+\frac{1.3.3}{2.4}\frac{aa_1^2}{a_1^4}b_{2,1}$$
$$-\frac{1.3}{2.4.2}\frac{(2a_1^2-3a^2)}{a_1^4}a_1^2b_{2,2}-\frac{1.3}{2.4}\frac{aa_1^2}{a_1^4}b_{2,3}\Big\}e_1^2\cos(2nt+2\varepsilon-2\varpi_1) \quad [59]$$

$$+m_1\Big\{-\frac{aa_1}{2.8a_1^2}b_{2,2}-\frac{a_1^2}{2.4a_1^2}b_{2,3}-\frac{1.3.9}{2.4.4}\frac{a^2a_1^2}{a_1^4}b_{2,1}+\frac{1.3.3}{2.4}\frac{a_1^2a}{a_1^4}b_{2,2}$$
$$-\frac{1.3}{2.4.2}\frac{(2a_1^2-3a^2)}{a_1^4}a_1^2b_{2,3}\Big\}e_1^2\cos(3nt-n_1t+3\varepsilon-\varepsilon_1-2\varpi_1) \quad [60]$$

$$+m_1\Big\{-\frac{aa_1}{2.8a_1^2}b_{2,3}-\frac{1.3.9}{2.4.4}\frac{a^2a_1^2}{a_1^4}b_{2,2}$$

$$+\frac{1.3.3}{2.4}\frac{a_i^3 a}{a_i^5}b_{5,3}\Big\}e_i^2\cos(4nt-2n_it+4\varepsilon-2\varepsilon_i-2\varpi_i) \quad [61]$$

$$-m_,\frac{1.3.9}{2.4.4}\frac{a^3 a_i^3}{a_i^5}b_{5,3}e_i^2\cos(5nt-3n_it+5\varepsilon-3\varepsilon_i-2\varpi_i) \quad [62]$$

$$+m_,\Big\{\frac{27}{8}\frac{a}{a_i^3}-\frac{3}{2.4}\frac{a\,a_i}{a_i^3}b_{3,0}-\frac{a_i^3}{2.4\,a_i^3}b_{3,1}-\frac{aa_i}{2.8\,a_i^3}b_{3,2}-\frac{1.3.2}{2.4}\frac{a_i^3 a}{a_i^5}b_{5,0}$$
$$-\frac{1.3}{2.4.4}\frac{(4a_,^2-5a^2)}{a_i^3}a_,^2\,b_{3,1}+\frac{1.3.3}{2.4}\frac{aa_i^3}{a_i^5}b_{5,2}$$
$$-\frac{1.3.9}{2.4.4}\frac{a^3 a_i^3}{a_i^5}b_{5,3}\Big\}e_i^2\cos(nt-3n_it+\varepsilon-3\varepsilon_,+2\varpi_i) \quad [63]$$

$$+m_i\Big\{-\frac{3}{2.8}\frac{aa_i}{a_i^3}b_{3,1}-\frac{a_i^3}{2.4\,a_i^3}b_{3,2}-\frac{aa_i}{2.8\,a_i^3}b_{3,3}-\frac{1.3}{2.4.2}\frac{a^3 a_i^3}{a_i^5}b_{5,0}$$
$$-\frac{1.3}{2.4}\frac{aa_i^3}{a_i^5}b_{5,1}-\frac{1.3}{2.4.2}\frac{(2a_i^2-3a^2)}{a_i^5}a_,^2\,b_{5,2}$$
$$+\frac{1.3.3}{2.4}\frac{aa_i^3}{a_i^5}b_{5,3}\Big\}e_i^2\cos(2nt-4n_it+2\varepsilon-4\varepsilon_i+2\varpi_i) \quad [64]$$

$$+m_i\Big\{-\frac{3}{2.8}\frac{aa_i}{a_i^3}b_{3,2}-\frac{a_i^3}{2.4\,a_i^3}b_{3,3}-\frac{1.3}{2.4.4}\frac{a^3 a_i^3}{a_i^5}b_{5,1}-\frac{1.3}{2.4}\frac{aa_i^3}{a_i^5}b_{5,2}$$
$$-\frac{1.3}{2.4.2}\frac{(2a_i^2-3a^2)}{a_i^5}a_,^2\,b_{5,3}\Big\}e_i^2\cos(3nt-5n_it+3\varepsilon-3\varepsilon_i+2\varpi_i) \quad [65]$$

$$+m_i\Big\{-\frac{3}{2.8}\frac{aa_i}{a_i^3}b_{3,3}-\frac{1.3}{2.4.4}\frac{a^3 a_i^3}{a_i^5}b_{5,2}$$
$$-\frac{1.3}{2.4}\frac{aa_i^3}{a_i^5}b_{5,3}\Big\}e_i^2\cos(4nt-6n_it+4\varepsilon-6\varepsilon_i+2\varpi_i) \quad [66]$$

$$-m_i\frac{1.3}{2.4.4}\frac{a^3 a_i^3}{a_i^5}b_{5,3}e_i^2\cos(5nt-7n_it+5\varepsilon-7\varepsilon_i+2\varpi_i) \quad [67]$$

$$+m_i\Big\{\frac{a}{a_i^3}\Big(1+\frac{3}{2}e_,^2\Big)-\frac{aa_i}{a_i^3}b_{3,0}\Big\}\sin^2\tfrac{i_i}{2}\cos(nt+n_it+\varepsilon+\varepsilon_i-2\nu_i) \quad [68]$$

$$-m_,\frac{a\,a_i}{2\,a_i^3}b_{3,1}\sin^2\tfrac{i_i}{2}\cos(2n_it+2\varepsilon_,-2\nu_i) \quad [69]$$

$$m\frac{a\,a_i}{2\,a_i^3}b_{3,1}\sin^2\tfrac{i_i}{2}\cos(2nt+2\varepsilon-2\nu_i) \quad [70]$$

$$-m_,\frac{a\,a_i}{2\,a_i^3}b_{3,2}\sin^2\tfrac{i_i}{2}\cos(nt+3n_it+\varepsilon-3\varepsilon_,+2\nu_i) \quad [71]$$

$$-m_{,}\frac{a\,a_{,}}{2\,a_{,}^{2}}\,b_{3,2}\sin^{2}\tfrac{i_{,}}{2}\cos(3nt-n_{,}t+3\varepsilon-\varepsilon_{,}-2\nu_{,}) \qquad [72]$$

$$-m_{,}\frac{a\,a_{,}}{2\,a_{,}^{2}}\,b_{3,3}\sin^{2}\tfrac{i_{,}}{2}\cos(2nt-4n_{,}t+2\varepsilon-4\varepsilon_{,}+2\nu_{,}) \qquad [73]$$

$$-m_{,}\frac{a\,a_{,}}{2\,a_{,}^{2}}\,b_{3,3}\sin^{2}\tfrac{i_{,}}{2}\cos(4nt-2n_{,}t+4\varepsilon-2\varepsilon_{,}-2\nu_{,}) \qquad [74]$$

In the development of R, I have supposed $\iota = 0$, so that $\iota_{,}$ is the angle contained between the orbits of the planets P and $P_{,}$, or P_1 and P_2; in the general case, when ι_1, and ι_2, are retained, $\cos\iota_{,} = \cos\iota_1\cos\iota_2 + \sin\iota_1\sin\iota_2\cos(\iota_1 - \iota_2)$, ι_1 and ι_2, being the inclinations of the orbits of the planets P_1, P_2 to any plane $x\,y$, of which the direction is arbitrary,

$$r'r_{,}'\sin(\lambda' - \lambda_{,}') = r_1 r_2\Big\{\cos^2\tfrac{i_1}{2}\cos^2\tfrac{i_2}{2}\sin(\lambda_1 - \lambda_2)$$

$$-\sin^2\tfrac{i_1}{2}\cos^2\tfrac{i_2}{2}\sin(\lambda_1 + \lambda_2 - 2\nu_1) + \sin^2\tfrac{i_2}{2}\cos^2\tfrac{i_1}{2}\sin(\lambda_1 + \lambda_2 - 2\nu_2)$$

$$-\sin^2\tfrac{i_1}{2}\cos^2\tfrac{i_2}{2}\sin(\lambda_1 - \lambda_2 - 2\nu_1 + 2\nu_2)\Big\}$$

$$r'r_{,}'\Big\{\cos(\lambda' - \lambda_{,}') + s\,s_{,}\Big\} = r_1 r_2\Big\{\cos^2\tfrac{i_1}{2}\cos^2\tfrac{i_2}{2}\cos(\lambda_1 - \lambda_2)$$

$$+\sin^2\tfrac{i_1}{2}\cos^2\tfrac{i_2}{2}\cos(\lambda_1 + \lambda_2 - 2\nu_1) + \sin^2\tfrac{i_2}{2}\cos^2\tfrac{i_1}{2}\cos(\lambda_1 + \lambda_2 - 2\nu_2)$$

$$+\sin^2\tfrac{i_1}{2}\sin^2\tfrac{i_2}{2}\cos(\lambda_1 - \lambda_2 - 2\nu_1 + 2\nu_2)$$

$$+\sin\iota_1\sin\iota_2\cos(\lambda_1 - \lambda_2 - \nu_1 + \nu_2) - \sin\iota_1\sin\iota_2\cos(\lambda_1 + \lambda_2 - \nu_1 - \nu_2)\Big\}$$

Errata.

In page 330, *for* ι and ν, *read* $\iota_{,}$ and $\nu_{,}$.

$$-m\frac{\mu\mu'}{a'}\beta_2\sin\frac{i}{2}\cos\frac{i'}{2}\cos(2nt-n't+2\varepsilon'-\varepsilon-\varpi), \quad [25]$$

$$-m\frac{\mu\mu'}{a'}\beta_2\sin\frac{i}{2}\sin\frac{i'}{2}\cos(2nt-(n'+2)t+2\varepsilon'-\varepsilon+2\varpi), \quad [26]$$

$$-m\frac{\mu\mu'}{a'}\beta_2\sin\frac{i}{2}\sin\frac{i'}{2}\cos(2nt-2n't+2\varepsilon'-\varepsilon-2\varpi). \quad [27]$$

In the development of Ω, I have put $\varepsilon = 0$, so that γ is the angle contained between the orbits of the planets P and P', or P_1 and P_1'. In the general case, where γ_1 and ε_1 are retained, $\cos\gamma = \cos\gamma_1\cos\varepsilon + \sin\gamma_1\sin\varepsilon\cos(\varepsilon_1 - \varepsilon)$, ε_1 and ε_1' being the inclinations of the orbits of the planets P_1, P_1' to any plane xy at which the direction is arbitrary.

$$\gamma'\sin(\zeta'-\chi') = \gamma_1'\sin\gamma_1\left\{\cos\frac{\varepsilon}{2}\cos\frac{\varepsilon_1'}{2}\sin(\zeta_1'-\chi_1')\right.$$

$$-\sin\frac{\varepsilon}{2}\cos\frac{\varepsilon_1'}{2}\sin(\zeta_1'+\chi_1'-2\chi_1)+\sin\frac{\varepsilon}{2}\sin\frac{\varepsilon_1'}{2}\cos(\zeta_1'+\chi_1'-2\zeta_1)$$

$$\left.-\sin\frac{\varepsilon}{2}\cos\frac{\varepsilon_1'}{2}\sin(\zeta_1'-\chi_1'-2\chi_1+2\zeta_1)\right\}$$

$$+\gamma_1'\cos\gamma_1\left\{\sin(\chi_1'-\chi_1)+\varepsilon\,\varepsilon_1'\right\}=\gamma_1'\varepsilon_1'\left\{\cos\frac{\varepsilon}{2}\cos\frac{\varepsilon_1'}{2}\cos(\zeta_1'-\chi_1')\right.$$

$$+\sin\frac{\varepsilon}{2}\cos\frac{\varepsilon_1'}{2}\cos(\zeta_1'+\chi_1'-2\zeta_1)+\sin\frac{\varepsilon}{2}\sin\frac{\varepsilon_1'}{2}\cos(\zeta_1'+\chi_1'-\chi_1)$$

$$+\sin\frac{\varepsilon}{2}\sin\frac{\varepsilon_1'}{2}\cos(\zeta_1'-\chi_1'-2\chi_1+2\zeta_1)$$

$$\left.+\sin\varepsilon_1'\sin(\zeta_1'-\chi_1'+\varepsilon)-\sin\varepsilon\sin\varepsilon_1'\cos(\zeta_1'+\chi_1'-2\zeta_1)\right\}$$

In page 290, for ε and ε' read ε_1 and ε_1'.

XXIII. *On the Error in Standards of Linear Measure, arising from the thickness of the bar on which they are traced. By Captain* Henry Kater, *V.P. and Treasurer of the Royal Society.*

Read June 17th, 1830.

In the course of the adjustment and verification of the copies of the Imperial standard yard, destined for the Exchequer, Guildhall, Dublin, and Edinburgh, I discovered a source of error till then, I believe, wholly unsuspected, arising from the thickness of the bar upon the surface of which measures of linear dimension are traced. The difficulties which I experienced, and the remedy which suggested itself upon that occasion, and which was found efficient, are shortly detailed in the account of the construction and adjustment of the new standards of weights and measures of the United Kingdom of Great Britain and Ireland, published in the Philosophical Transactions for 1826.

But as the notice there given occupies little more than a single page, and might therefore pass unremarked, I cannot but think that a fact of such importance in inquiries where linear measures are concerned, and which may be sufficient to account for the discrepancies to be found in the experiments of different observers, ought to be placed before the Royal Society in a more prominent point of view than that which it at present occupies. I shall, therefore, first extract from the paper alluded to the part to which I refer, and then add an account of such experiments as I have since made on the subject; and describe a scale which I have caused to be constructed so as almost entirely to obviate the source of error of which I am treating.

It may be seen in the paper before mentioned, "On the Construction of the Standard Measures," that Sir George Shuckburgh's scale was employed as the medium of comparison; the distance from the zero point of which to that marked thirty-six inches, had been found by comparisons detailed in the Phi-

losophical Transactions for 1821, not to differ sensibly from the Imperial standard yard.

The bars intended for the Exchequer, &c. were half an inch thick and one inch and a quarter wide; the thickness being nearly the same as that of Sir GEORGE SHUCKBURGH's scale.

The dots upon the surface of the bars having been adjusted, so that their distance appeared to be equal to the distance from zero to thirty-six inches on Sir GEORGE SHUCKBURGH's scale, further comparisons were made after the plugs carrying the dots had been securely fixed; "and it was with surprise and disappointment that I found the whole of them apparently too short. They had been adjusted upon a board of mahogany carefully planed; and the table upon which they were now placed was so flat as to occasion little alteration in a spirit-level passed along it. The error of the standards was, however, far too considerable to be attributed to any curvature which on this occasion could take place; and it was not until after several days that I discovered the cause of this perplexing circumstance. I found that by placing a card, the thickness of which was accurately one-fiftieth of an inch, under the middle of the standard, the distance of the dots was much increased; and by placing a card of the same thickness under each of the extremities, and withdrawing that which was under the centre, the distance of the dots was considerably diminished. The total difference amounted to no less than .0016 of an inch; whilst the double of the error, which would have arisen from mere curvature under similar circumstances, would not have been one ten-thousandth of an inch.

"The cause was now evident: by elevating the middle of the standard, the under surface was shortened and the upper surface extended; and on the contrary, when the extremities were elevated, the upper surface was compressed and the lower surface lengthened; the quantity of the effect evidently depending upon the thickness of the bar.

"Having thus assured myself of the source of the error, a method of obviating it soon presented itself. As the upper and under surfaces of the bar are in different states, the one being compressed and the other extended, there must be an intermediate plane which suffers neither extension nor compression; and this plane must be nearly midway between the two surfaces. I therefore

caused Mr. Dollond to reduce the thickness of the bar for the distance of an inch and three-quarters from its extremities to one-half; the gold disks and plugs were then inserted as before, and the adjustment completed in the manner which has been described. The plugs being secured and the projecting parts removed, the standards were repeatedly compared with Sir George Shuckburgh's scale (the standard being placed upon the scale), when no perceptible difference could be detected. Pieces of card were now placed under the standard as before, without occasioning any appreciable alteration; and I had thus experimental proof of the perfect efficiency of the remedy I had employed*."

It will now be necessary for me to state some circumstances which led to the experiments I am about to detail.

In the year 1820, I very carefully compared a scale, now in the possession of Mr. Dollond, with Sir George Shuckburgh's standard, when the distance from zero to thirty-six inches upon both scales appeared to be precisely the same. In the year 1824, I again compared Mr. Dollond's scale with Sir George Shuckburgh's, upon the occasion of examining a scale for M. Schumacher and another for M. Svanberg. Mr. Dollond's scale by these comparisons appeared to differ from Sir George Shuckburgh's only .000181 of an inch in defect. I had thus good reason to suppose the value of Mr. Dollond's scale to be well ascertained; and as it was much thinner than that of Sir George Shuckburgh's, I considered it preferable for determining the value of the scales intended for Denmark and Sweden.

Having recently received from Mr. Dollond a scale which I had instructed him to make for the Government of Russia, and also the Imperial standard yard from the House of Commons, I commenced the necessary preparations for comparing them together.

Finding by stretching a thread along the table, which I have always used for such comparisons, that its surface was concave, I had it carefully planed until the thread no longer indicated any irregularity. My first comparisons at once showed a considerable difference between the scale intended for Russia (copied from Mr. Dollond's scale) and the Imperial standard yard; at the same time that a scale which I had formerly laid off for my own use from Mr. Dollond's,

* Phil. Trans. for 1826, Part II. p. 44–46.

and which might be taken as equal to it, differed very little from the Russian scale. I now procured Sir GEORGE SHUCKBURGH's scale and Mr. DOLLOND's, and set myself carefully to investigate the cause of this difference.

It was immediately evident that Sir GEORGE SHUCKBURGH's scale still nearly agreed with the Imperial standard yard, and that there was no great difference between Mr. DOLLOND's scale, my scale, and that intended for Russia. But the difference between Mr. DOLLOND's scale and Sir GEORGE SHUCKBURGH's was so great, as to leave no doubt on my mind that in determining the value of the former, I had committed some considerable error; and as other scales are dependent upon this, it became an object of importance to ascertain its origin and amount. The origin of this error I could not for a moment hesitate in pronouncing to be the thickness of Sir GEORGE SHUCKBURGH's scale; the very great care I have been accustomed to bestow in comparisons of this kind, leaving me no other cause to which it could with probability be referred.

I have said that a thread stretched along the table indicated no irregularity of surface. I now examined the surfaces of the Imperial standard yard by stretching a thread along that surface which was perpendicular to the table. Every face of this bar was perceptibly concave; yet when laid upon the table and pressed horizontally at one end, it moved about its centre, proving that the surface of the table was convex, though the thread was not capable of indicating it.

I next procured a marble slab nearly sixty-four inches long. This I preferred in its unpolished state; as the operation of polishing being performed upon small portions of the marble in succession might destroy the plane surface procured by grinding. Upon this slab I placed the Imperial standard yard, Sir GEORGE SHUCKBURGH's scale, my own, and Mr. DOLLOND's scale; the Russian scale being laid, for want of room, upon the table. The Imperial standard yard seemed now to rest with nearly its whole surface in contact with the marble, and this, in addition to the test of the thread, I considered to be a sufficient indication that the marble was plane.

The following comparisons were then made between the different scales, in which the same microscopes and apparatus were employed as are described in an account of the comparison of various British standards of linear measure, published in the Philosophical Transactions for 1821. It is to be observed that

as the microscopes invert, an increase in the readings of the micrometer indicates that the scale is shorter than that with which it is compared, and vice versâ. The value of one division of the micrometer is .0000428742 of an inch.

Date. 1830.	Therm.	Imperial Standard Yard.	Sir George Shuckburgh's Scale.	Russian Scale.	Mr. Dollond's Scale.	Captain Kater's Scale.
	°	div.	div.	div.	div.	div.
May 1	66	7	0.5	23	27.5	32.7
		8.2	2.0	23.7	25	30
2	63	104	98	120	124.5	126
		76.5	72.5	95	101	101
		8.7	5	22.5	28.5	37
	66	88	80.5	95	104	110.5
3	62	3.5	2	23	27	28.2
		3	0	24.5	26	....
		102.5	96	117	125	131
		17	14	35.5	42	46
	64	10.7	4	22.5	28.3	37
4	61	20.2	16	39	40	51
		20	16.5	37	43.6	47.5
	62	18	16	34.6	42	46
5	63	15	10.7	28	37	37.7
		14	10	26.7	33.5	42
Means...		32.02	27.73	47.94	53.43	60.24
Readings of Imperial Stand. Yd.			32.02	32.02	32.02	32.02
Differ. from Imperial Stand. Yd.			4.29	15.92	21.41	28.22

Converting these differences of the readings of the micrometer into inches, we obtain the distances from zero to thirty-six inches on each scale in parts of the Imperial standard yard.

Sir George Shuckburgh's scale . . .	36.00018 inches
The Russian scale	35.99932 ——
Mr. Dollond's scale	35.99908 ——
Captain Kater's scale	35.99879 ——

Here it is seen that Mr. Dollond's scale, which by careful comparisons in the years 1820 and 1824 appeared to differ little, if at all, from Sir George Shuckburgh's scale, is now no less than .0011 of an inch in defect.

I now resolved to determine the amount of the error which might arise from the flexure of a bar of known thickness, the curvature of the surface upon which it is placed being given.

For this purpose I prepared sets of wires, which by careful measurement I found to be, .012, .02, .03, .04, .05, and .10 of an inch in diameter.

In order to subject the surface of the marble slab to a more rigid examination, I prepared some fine copper wire by annealing it, and then stretched it along the slab, applying as much weight as it would bear at each end without breaking. This detected a curvature of one end of the slab, which the thread did not indicate, and which was removed by placing a thick card between it and the table.

The Imperial Standard Yard.

I commenced with the Imperial standard yard. The bar upon which this yard is laid off, is 1.07 inch square.

On placing a wire of .012 of an inch diameter under the middle of the bar, the two extremities were raised above the marble; a thin wedge was placed under one extremity of the bar, the other being then in contact with the marble, in order to prevent the effect which would arise from the weight of the apparatus carrying the microscopes when placed upon the bar. This precaution, I shall content myself with remarking, was used in every instance where necessary.

The following comparisons were then made between the length of the yard when the bar lay flat upon the marble, when it was curved upwards by a wire placed under its middle, and when curved downwards by wires placed under the dots marking the yard. In the latter position the middle of the bar touched the marble.

With wires of .012 of an inch diameter.

	Without Wires.	A Wire under the middle.	Wires under the ends.	Error when curved upwards.	Error when curved downwards.	Sum of Errors.
	div.	div.	div.	div.	div.	div.
	93.5	89.5	112.0			
	96.5	91.7	111.5			
(A.)	95.7	88.0	111.7			
	79.0	72.5	94.5			
	Mean 91.7	85.4	107.4	6.5	15.5	22.0
	0.2 Reduction to the Chord.					
	91.9					

As in the preceding experiments the bar was clear of the marble when a wire was placed under its middle, and touched the slab when wires were placed under its extremities; I next tried whether any greater errors would be produced by the curvature which the bar would assume by its own weight when clear of the marble in both cases. For this purpose I used wires of .05 of an inch diameter, by which the bar was wholly supported.

With wires of .05 of an inch diameter.

(B.)

Without Wires.	A Wire under the middle.	Wires under the ends.	Error when curved upwards.	Error when curved downwards.	Sum of Errors.
div.	div.	div.	div.	div.	div.
67.0	60.0	79.0			
61.0	54.0	78.5			
57.5	54.0	74.5			
60.0	52.0	73.0			
57.0	55.5	74.5			
Mean 60.5	55.1	75.9	5.6	15.2	20.8
0.2 Reduction to the Chord.					
60.7					

Here it may be seen that the results are very nearly the same as in the preceding Table, and consequently that the errors of the Imperial standard yard are the greatest possible with a curvature, the versed sine of which is less than one-hundredth of an inch.

In order to obtain accurate conclusions from the above Tables, it must be considered that when the bar is lying flat upon the marble it is the length of the scale that is taken, but when the bar is curved it is the chord of the arc which is measured. It therefore becomes necessary to add to the mean of the column headed "without wires" the number of divisions of the micrometer, which is equal to the excess of the arc above its chord.

This correction being made, if the difference between the mean of the readings so corrected of the micrometer, when the bar lies flat upon the marble, and the mean of the readings when a wire is placed under the middle of the bar be taken, it will give the amount of the error arising from flexure independent of the chord when the bar is curved upwards.

In like manner the difference between such corrected mean and the mean of

the readings of the column headed "wires under the ends" will give the error proceeding from the same source when the bar is curved downwards: and the sum of the errors thus obtained will show the greatest error which can arise from a curvature, the versed sine of which is equal to the diameter of the wires employed. The same amount of errors may at once be obtained by taking the difference of the mean readings of the second and third columns.

In the Imperial standard yard it appears that with a curvature, the versed sine of which is less than .012 of an inch, the amount of the errors is .000943, or nearly one-thousandth of an inch; whilst the error which would result from the difference between the arc and its chord is absolutely insensible, not amounting to one hundred-thousandth of an inch. Now it must be obvious, that if a scale were compared at two different periods with the Imperial standard yard, the yard at one of such periods being placed upon a part of the table deviating from a plane surface .012 of an inch in a yard, and at another period the same quantity, but in a contrary direction,—the difference in the resulting values of the scale so compared would be no less than .000943 of an inch. This supposes the scale which is the subject of comparison either to be very thin, or to be placed upon a part of the table which is perfectly flat; a circumstance which it is not difficult to imagine very possible, or even that different parts of the surface of the table may be curved in contrary directions, when the small amount of the curvature in question is considered.

I may here observe, that the thickness of a single shaving which the plane takes off from the table is sufficient to occasion an error equal to that resulting from the preceding comparisons; for I found by careful measurements with a micrometer, that the mean thickness of such a shaving of deal was about .009 of an inch.

I have hitherto supposed the surface of the table not to be plane; but if the table were plane and the surface of the bar were curved, the bar would by its weight apply itself to the plane surface of the table; and a like error in either case would be the consequence.

It has been stated that the surface of the bar of the Imperial standard yard is concave; and by my present comparisons the distance from zero to thirty-six inches on Sir GEORGE SHUCKBURGH's scale, which by former measurements

appeared not to differ sensibly from the Imperial standard yard, is in excess .0001945, or nearly two ten-thousandths of an inch. Now if the surface of the marble slab be supposed to be plane, the curvature of the Imperial standard yard would have the same effect in producing error as in the case of wires being placed under its ends; and it will be seen that a curvature of the bar, the versed sine of which is only .002 of an inch in length, would be sufficient to produce the difference in question.

On examining the preceding Table, it will be perceived that the difference between the length of the yard when lying flat on the marble, and its length when curved upwards, is much less than the difference when the bar is supported at its ends, and consequently curved downwards; and hence that the neutral plane is not as I supposed in the middle of the bar, but nearer to the convex surface: but on this important part of the subject I shall have to state more hereafter.

Sir George Shuckburgh's *Scale*.

The bar upon which this scale is traced is 67.7 inches long, 1.4 inch wide, and 0.42 inch thick. The scale comprises 60 inches, the distance from the end of the bar to zero is 3.7 inches, and from 60 to the other extremity of the bar 4 inches.

The scale being placed upon the marble slab, its ends projected two inches beyond it. I could perceive light between the bar and the marble, and could pass a sheet of letter-paper under the zero end to the distance of eight inches from its extremity, that is to the fourth inch of the scale. The bar was in contact with the marble for two inches and a quarter at the other end; and from this part letter-paper could be passed between the bar and the marble for the distance of eighteen inches. The remainder of the bar rested upon the marble.

In the following experiments, the wires in elevating the extremities were placed under 0 and 60 of the scale, and under 30 when the scale was to be curved upwards.

With wires .012 of an inch diameter.

From 0 to 36 inches.

	Without Wires.	A Wire under the middle.	Wires under the ends.	Error when curved upwards.	Error when curved downwards.	Sum of Errors.
	div.	div.	div.	div.	div.	div.
(C.)	99.0	95.5	105.0			
	99.0	95.5	106.0			
	Mean 99.0	95.5	105.5	3.5	6.5	10.0
	0.0 Reduction to the Chord.					
	99.0					

With wires .02 of an inch diameter.

	Without Wires.	A wire under the middle.	Wires under the ends.	Error when curved upwards.	Error when curved downwards.	Sum of Errors.
	div.	div.	div.	div.	div.	div.
	6.0	3.0	14.0			
	3.0	0.5	17.0			
(D.)	97.0	92.0	109.3			
	96.0	94.0	109.0			
	Mean 50.5	47.4	62.3	3.1	11.8	14.9
	0.2 Reduction to the Chord.					
	50.7					

With wires .03 of an inch diameter.

	Without Wires.	A Wire under the middle.	Wires under the ends.	Error when curved upwards.	Error when curved downwards.	Sum of Errors.
	div.	div.	div.	div.	div.	div.
	97.0	92.7	110.2			
(E.)	96.5	92.0	111.0			
	Mean 96.7	92.3	110.6	4.8	13.5	18.3
	0.4 Reduction to the Chord.					
	97.1					

With wires .04 of an inch diameter.

	Without Wires.	A Wire under the middle.	Wires under the ends.	Error when curved upwards.	Error when curved downwards.	Sum of Errors.
	div.	div.	div.	div.	div.	div.
	16.0	7.5	32.0			
(F.)	12.0	8.0	30.7			
	94.5	87.0	114.0			
	Mean 40.6	34.2	58.9	7.2	17.5	24.7
	0.8 Reduction to the Chord.					
	41.4					

Wishing to ascertain whether the curvatures of both halves of the scale were similar, I took the distance from 24 to 60 inches,

With wires .04 of an inch diameter.

From 24 to 60 inches.

(G.)

Without Wires.	A Wire under the middle.	Wires under the ends.	Error when curved upwards.	Error when curved downwards.	Sum of Errors.
div.	div.	div.	div.	div.	div.
41.0	24.0	44.0			
38.0	21.0	44.0			
126.0	104.0	129.0			
Mean 68.3	49.7	72.3	19.4	3.2	22.6
0.8 Reduction to the Chord.					
69.1					

Here we perceive that the sum of the errors is very nearly the same as before; but there is an irregularity in the curvature, or rather a difference in the place of the neutral plane of the two halves of the bar, which deserves particular attention; for instead of the error being less, as in all the preceding experiments, when a wire is placed under the middle, than it is when the extremities of the bar are supported, it is here greater. We may also remark that the difference between the means of the readings "without wires" is 27.7 divisions, which shows that the distance from 24 to 60 inches of the scale is shorter than that from 0 to 36 inches .001187 of an inch.

With wires .05 of an inch diameter.

From 0 to 36 inches.

(H.)

Without Wires.	A Wire under the middle.	Wires under the ends.	Error when curved upwards.	Error when curved downwards.	Sum of Errors.
div.	div.	div.	div.	div.	div.
14.5	7.5	39.5			
13.0	3.5	33.0			
11.5	4.0	31.2			
8.5	4.0	36.7			
7.0	0.0	32.0			
Mean 10.9	3.8	34.5	8.3	22.4	30.7
1.2 Reduction to the Chord.					
12.1					

With wires 0.5 of an inch diameter.

From 24 to 60 inches.

	Without Wires.	A Wire under the middle.	Wires under the ends.	Error when curved upwards.	Error when curved downwards.	Sum of Errors.
	div.	div.	div.	div.	div.	div.
	41.5	23.0	48.0			
	39.5	18.5	50.0			
	39.5	19.0	49.0			
(I.)	38.7	15.7	48.5			
	35.0	16.0	45.0			
	Mean 38.8	18.4	48.1	21.6	8.1	29.7
	1.2 Reduction to the Chord.					
	40.0					

From this Table the same conclusions may be drawn as from Table (G), as to the place of the neutral plane in the two halves of the bar. I am disposed to attribute this irregularity to the manner in which the scale is constructed, as it consists of a bar of cast brass, to which is soldered a piece of plate brass, upon which the divisions are traced. The difference between the distance from 24 to 60 inches and from 0 to 36 inches does not differ sensibly from that before found, namely .001187 of an inch. I should have observed that the comparisons of the distance from 0 to 36 and from 24 to 60 inches on the scale immediately followed each other, that is, the microscopes were transferred immediately from one part of the scale to the other.

Wishing to try the effect of an increased curvature, the following comparisons were made.

With wires 0.1 of an inch diameter.

From 12 to 48 inches.

	Without Wires.	A Wire under the middle.	Wires under the ends.	Error when curved upwards.	Error when curved downwards.	Sum of Errors.
	div.	div.	div.	div.	div.	div.
	81.0	51.5	121.0			
	83.5	51.0	117.0			
(K.)	86.0	53.0	120.0			
	83.0	51.0	120.0			
	Mean 83.4	51.6	119.5	36.8	31.1	67.9
	5.0 Reduction to the Chord.					
	88.4					

The above Table shows that employing the middle portion of the scale, the error in either position is nearly the mean of the errors in the two preceding Tables.

Mr. DOLLOND'S *Scale.*

This scale is upon plate brass, 42 inches long, 1.6 inch wide, and 0.17 inch thick; it contains 41 inches. The scale when placed upon the marble slab appeared to be in contact with it throughout its whole extent.

The wires for elevating the ends were placed under 0 and 41 inches, and the wire when the middle was raised was under 20½ inches.

With wires .05 of an inch diameter.

From 0 to 36 inches.

	Without Wires.	A Wire under the middle.	Wires under the ends.	Error when curved upwards.	Error when curved downwards.	Sum of Errors.
	div.	div.	div.	div.	div.	div.
	32.5	30.0	52.0			
	32.0	30.0	52.5			
	32.0	30.0	56.0			
(L.)	31.8	31.2	57.0			
	30.0	28.0	52.6			
	30.3	28.0	52.6			
	Mean 31.4	29.5	53.8	4.4	19.9	24.3
	2.5 Reduction to the Chord.					
	33.9					

Captain KATER'S *Scale.*

This scale is of cast brass, 45.2 inches long, ⅝ wide, and 0.29 inch thick. The wires, when the ends were elevated, were placed under the extremities.

With wires 0.5 of an inch diameter.

From 0 to 36 inches.

	Without Wires.	A Wire under the middle.	Wires under the ends.	Error when curved upwards.	Error when curved downwards.	Sum of Errors.
	div.	div.	div.	div.	div.	div.
	105.0	97.0	137.0			
	89.0	82.5	128.0			
	92.5	80.5	123.0			
(M.)	91.0	79.0	128.0			
	95.0	80.0	128.5			
	94.0	80.0	131.0			
	Mean 94.4	83.2	129.3	13.2	32.9	46.1
	2.0 Reduction to the Chord.					
	96.4					

I shall now state the results which have been obtained from the comparisons I have detailed.

It has been already shown that a correction must be applied to the observed length of the scale, to reduce it to the chord of the given curvature. This, if the scale is only 36 inches long, may be applied at once; but with Sir GEORGE SHUCKBURGH's scale, the length of which is 60 inches, it becomes necessary first to find the difference between an arc of 60 inches and its chord, and then to take the proportional part of this difference for 36 inches. The same must be done with Mr. DOLLOND's scale, which is 41 inches long, and with my own, the length of which is 45.2 inches.

The following Table gives the difference between the arc and its chord, and the proportional part where necessary for 36 inches.

Versed Sine.	Difference of the Arc and its Chord on 36 inches.	Proportional part for 36 inches.	Equivalent Divisions of Micrometer.
inches.	inches.	inches.	
.012	.00001		
.02	.00002		
.03	.00006		
.04	.00009		2.1
.05	.00014		
Versed Sine.	**Difference of the Arc and its Chord on 60 inches.**		
.012	.000005	.000003	0.0
.02	.000013	.000008	0.2
.03	.000030	.000018	0.4
.04	.000054	.000032	0.8
.05	.000084	.000050	1.2
.10	.000400	.000240	5.0
Versed Sine.	**Difference of the Arc and its Chord on 41 inches.**		
.05	.000122	.000107	2.5
Versed Sine.	**Difference of the Arc and its Chord on 45.2 inches.**		
.05	.000112	.000089	2.0

Having applied the reduction to the chord, we obtain the error occasioned

by the extension of the surface of the bar when it is curved upwards, and the error when it is curved downwards; but in Sir GEORGE SHUCKBURGH's scale, when a wire is placed under the middle, the surface of the bar rests upon the edges of the marble slab; the length to be considered of the bar is therefore in this position equal to the length of the marble 63.7 inches, and the error of the curvature upwards, as given in the Tables, must be increased in the proportion of 63.7 to 60.

The same method must be pursued with Mr. DOLLOND's scale, the bar being forty-two inches long.

Lastly, these results must be reduced, but in inverse proportion, to what they would have been had the length of the scale been thirty-six inches.

As the distance from 0 to 36 inches is the part of Sir GEORGE SHUCKBURGH's scale, which has been considered upon every occasion as equal to the Imperial standard yard, it is this portion to which all the subsequent deductions refer.

The following Table contains the results of the foregoing experiments, with each scale reduced to a bar of thirty-six inches in the manner which has been described, the bar being taken as equal in thickness to each scale respectively.

	Thickness of Bar.	Table.	Versed Sine.	Error when curved upwards.	Error when curved downwards.	Sum of Errors.
	inches.		inches.	inches.	inches.	inches.
Imperial Standard Yard	1.07	(A)	.012	.00028	.00066	.00094
Sir GEORGE SHUCKBURGH's Scale.	0.42	(C)*	.01	.00022	.00039	.00061
		(D)	.02	.00023	.00084	.00107
		(E)	.03	.00036	.00096	.00132
		(F)	.04	.00054	.00125	.00179
		(H)	.05	.00063	.00160	.00223
Mr. DOLLOND's Scale	0.17	(L)	.05	.00022	.00097	.00119
Captain KATER's Scale	0.29	(M)	.05	.00071	.00176	.00247

By referring to the results in this Table, derived from Sir GEORGE SHUCKBURGH's scale, it will be perceived that the sum of the errors increases as the versed sine or diameter of the wire employed. This will be more readily shown by dividing the sum of the errors by the corresponding diameter of the wire, or reducing each to a versed sine of .01 of an inch.

* Reduced by proportion from a versed sine of .012 to that of .01 of an inch.

Versed Sine.	Sum of Errors.	Sum of Errors Reduced to a Versed Sine of .01 of an inch.
inches.	inches.	inches.
.01	.00061	.00062
.02	.00107	.00053
.03	.00132	.00044
.04	.00179	.00045
.05	.00223	.00045
	Mean ..	.00050

The errors arising from the thickness of different scales, though nearly in proportion to their thickness, do not seem to follow any very regular law, but to depend in some measure upon the manufacture of the material employed, as well as upon the thickness of the bar: thus if we reduce the error of each scale to what it would have been had the scale been half an inch thick, thirty-six inches long, and the versed sine equal to .01 of an inch, we have the following results for comparison.

Errors resulting from a varied sine of .01 of an inch, the bar being supposed thirty-six inches long, and half an inch thick.

	Error when curved upwards.	Error when curved downwards.	Sum of Errors with a Versed Sine of .01 of an inch.
	inches.	inches.	inches.
Imperial Standard Yard	.00013	.00031	.00044
Sir George Shuckburgh's Scale..	.00017	.00042	.00059
Mr. Dollond's Scale	.00012	.00056	.00068
Captain Kater's Scale	.00025	.00061	.00086

Here it may be seen that the sum of the errors of my scale is the greatest, and this scale is upon a bar of cast brass not hammered.

Mr. Dollond's scale is of plate brass, and Sir George Shuckburgh's appears to be made of cast brass faced with plate brass.

The Imperial standard yard is upon a bar of brass, which I should judge from its apparent hardness to have been well hammered; but the preceding result given from this bar must be too little, as it has been shown that a wire .012 of an inch diameter was more than sufficient to produce the greatest errors to which the bar is liable.

From the last Table it appears that the error of the scale when curved upwards is scarcely more than one half of its error when curved downwards; and from this it should seem that the neutral plane is not in the middle of the bar as I had supposed.

If this inference be correct, the distance of the neutral plane from that surface of the bar, the curvature of which is convex, may be found by considering the sum of the errors in the preceding Table to represent the thickness of the bar, when the corresponding error resulting from the curvature upwards will be the distance of the neutral plane from the convex surface. This distance appears to be scarcely equal to one-third of the thickness of the bar.

From the experiments which have been detailed, we are led to the following conclusions.

1. That in a standard of linear measure traced upon the surface of a bar, an error arises from the thickness of the bar when it is placed upon a table the surface of which is not plane.

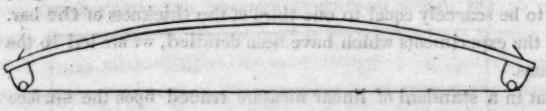
2. That this error in bars of the same material and of unequal thickness is within certain limits as the thickness of the bar, and depends upon the extension of that surface of the bar which becomes convex, and the compression of the surface which is concave.

3. That the error to which the same scale is liable from this cause, is directly as the versed sine of the curvature of the surface upon which the scale is placed.

4. That this error very far exceeds that which would arise from the difference of length between the arc and its chord under similar circumstances; so much so, that the sum of the errors from this cause in a bar one inch thick, with a versed sine of not one-hundredth of an inch, is nearly one-thousandth of an inch; whilst double the difference between the chord and the arc is not one fifty-thousandth.

It was not until I had written thus far, that a method, with which I am perfectly satisfied, occurred to me of trying a surface supposed to be plane. The difficulty, if not the impossibility of procuring what is called a straight edge is well known to workmen; but this desideratum I supplied by the very easy process of stringing a bow six feet long with piano-forte wire one-hundredth of an inch diameter, which bears a considerable degree of tension without breaking. The wire was passed over two thick wires half bedded in pieces glued

near the extremities of the bow; these served as bridges, and one was of a sufficient length to support the whole when laid upon a horizontal surface. The nature of the surface upon which it is placed may thus be examined at leisure by observing its proximity to the wire. Should the surface be convex, a wire must be placed under each extremity of the bow-string of sufficient diameter just to clear it from all but the most convex part of the surface. As this simple contrivance may be applied to a great variety of useful purposes where a straight edge is required, I trust I may be pardoned for giving the accompanying sketch of it.



By means of this apparatus I examined a variety of surfaces, any one of which I formerly should have considered as well calculated to serve as a support for scales, which were to be compared together. The following were the results.

	Length in inches.	Curvature.	Versed Sine inches.
	inches.		inches.
A mahogany dining-table	42	Concave.	0.04
A marble chimney-piece	61	Ditto	0.12
Another	62	Ditto	0.04
Another	62	Ditto	0.10
Top of a piano-forte of rosewood finely finished	48	Ditto	0.03

The front board of a piano-forte of rosewood forty-seven inches long. This was very highly finished, and the general surface was found to be nearly plane, but irregular. Here it is worthy of observation, that I could detect the nature and extent in some degree of the irregularities of the surface, by tapping with my fingers upon the wire whilst it was pressed by the weight of the bow upon the board. Where it yielded no sound, the wire was of course in contact with the surface, which in that case was either convex or plane. When the wire yielded a sound by tapping upon it with the finger, the surface was concave; and some idea might be formed of the extent, by the acuteness or gravity of the sound produced; the edges of the concavity serving as bridges, which limited

the length of the string. So delicate is this test by sound, that a concavity can be detected by this method where the interval between the wire and the surface under examination is not to be perceived by the eye.

Having now a means superior to any I before possessed of examining a plane surface, I applied it to the marble slab, and detected a slight concavity not exceeding one-hundredth of an inch. This was corrected by placing cards under different parts of the marble. When I had adjusted it so that no irregularity was perceptible, I thought it necessary to repeat my former comparisons.

The marble most carefully made plane by the bow.

Date. 1830.	Imperial Standard Yard.	Sir George Shuckburgh's Scale.	Russian Scale.	Mr. Dollond's Scale.
	div.	div.	div.	div.
May 31	7	5.5	26	31.0
	4.5	6.2	25	33.0
	4.7	7.0	24	33.2
	4.7	4.8	23.2	32
	3.8	3.3	20	30
June 1	4.0	6.0	22	31.2
	7.0	4.0	22	30.0
	5.0	4.0	21	32
2	10	6.0	23.5	31.3
	8	3.0	25	31
	6.3	3.0	22	32.5
	6	4.5	21.2	30
	3	3.5	23.5	31
	7	4	21.5	30.5
3	98.0	97	114.5	122.3
	97.5	96.5	115	123.5
4	28	22	43	53
	27.5	20	40.5	51
5	38.0	33.5	53	59.5
	37.3	31	50	59.3
Mean	20.34	18.24	36.79	45.36
Readg of Impl Stand. Yd	20.34	20.34	20.34	20.34
Diff. from Impl Stand. Yd		2.10	16.45	25.02

From these differences converted into inches, we have now the following distances from 0 to 36 inches on each scale in parts of the Imperial standard yard.

Sir George Shuckburgh's scale . . . . 36.00009 inches.
The Russian scale . . . . . 35.99929 ——
Mr. Dollond's scale . . . . . 35.99893 ——

These results differ but little from those of the former comparisons, to which, however, I think they are to be preferred.

It has been shown that from the present results, the value which has been hitherto given to Mr. DOLLOND's scale requires a considerable correction. Mr. DOLLOND's scale, by former measurements, appeared to be equal to 35.99991 inches of Sir GEORGE SHUCKBURGH's scale, and consequently to be shorter than that scale only .00009 of an inch; but it now appears to be shorter than Sir GEORGE SHUCKBURGH's scale .00116 of an inch. Sir GEORGE SHUCKBURGH's scale, therefore, when the former comparisons were made, must have been curved downwards in consequence of that part of the surface of the table upon which it was placed being concave; and we may remark that by consulting the preceding Tables, it will be seen that a curvature, the versed sine of which is .03 of an inch in a yard, would have been sufficient to occasion the error in question.

With reference to this error in the former estimation of the value of Mr. DOLLOND's scale, it is important to add that in the year 1828 I employed it (not having then access to Sir GEORGE SHUCKBURGH's standard) in determining the value of a scale for the Government of Hanover. This scale was made by Mr. DOLLOND, according to the mode which will hereafter be described; and when referred to Sir GEORGE SHUCKBURGH's scale, then taken as equivalent to the Imperial standard yard, it appeared to be equal to 35.99973 inches: upon that occasion the scale for Hanover was found by numerous comparisons to be shorter than Mr. DOLLOND's scale .00018 of an inch, which being subtracted from 35.99893 inches, the last determination of the value of Mr. DOLLOND's scale, leaves for the true length of the Hanoverian scale 35.99875 inches of the Imperial standard yard.

Having now shown the nature and magnitude of the error, which is the subject of this paper, I shall proceed to point out the means of obviating it.

It has been seen that the error in question results from the extension and compression of the surfaces of the bar upon which the scale is laid off dependent upon its curvature, and that there is a neutral surface which suffers neither extension nor compression, and which appears from the preceding experiments to be at about one-third of the thickness of the bar from that surface which becomes convex. When it is the object to have two points only on the

bar, marking for example the yard, it has been already shown that by cutting away one-half of the thickness of the bar at its ends and placing the points upon the new surface, the error which arises from flexure is reduced to the least possible quantity; as this (the difference between one-half and one-third of the thickness of the bar) is the nearest approach that can be made to the neutral surface when the bar is curved upwards and when it is curved downwards.

When a scale of inches is required, this method is not available; as the whole surface of the bar must in that case be employed. But by diminishing the thickness of the bar, the magnitude of the error is diminished proportionably; and it is evident that the thickness of the bar might be so reduced as to render the error scarcely appreciable.

Having prepared a bar, or rather a plate, of a thickness which is no more than sufficient to receive the divisions of the scale, and to preserve an even surface, the next object is to provide this plate with a proper support. For this purpose a bar of brass well hammered is to be employed, of a sufficient width, and half an inch or even an inch thick. Upon the surface of this bar the thin plate intended to receive the divisions is to be placed, and made to slide freely in a dovetailed groove, formed by two side-plates of similar thickness, screwed to the surface of the bar. Lastly, the thin plate upon which the scale is to be laid off is to be fixed at its middle point to the bar by a single screw passing through it.

Now if the thick bar which forms the support be curved, its surfaces will suffer extension and compression in proportion to the thickness of the bar. The thin plate accommodating itself to the curvature of the bar will follow the same law, and the resulting error will be less in the proportion of the thickness of the plate to that of the bar. Now it has been shown that the greatest errors to which the Imperial standard yard (one inch thick) is liable, amount to nearly one-thousandth of an inch; as a wire of one-hundredth of an inch diameter placed under the middle of the bar is more than sufficient to produce the greatest curvature of which it is susceptible; and it follows that if such a bar were to form the support of a plate of one-tenth of an inch in thickness in the manner just described, the sum of the greatest errors, to which a scale

so constructed would be liable from curvature, could not exceed one ten-thousandth of an inch.

The support of the scale which I have caused to be made for the Government of Russia, and the comparisons of which, with the Imperial standard yard, are given in the present paper, is a brass bar one inch and three-quarters wide, and half an inch thick. The plate carrying the divisions is one-tenth of an inch thick; but were I to construct another, it should not exceed half this thickness. This plate is made to slide freely upon the surface of the bar in a dovetailed groove, and is fixed to the bar at its middle point by a single screw in the manner before described. The divisions are thirty-six inches, marked by dots upon the surfaces of gold pins let into the brass plate. An additional inch beyond zero is divided in like manner into tenths of an inch. This description will serve also for that of the scale before alluded to, made for the Government of Hanover.

The following experiments were made with the Russian scale, the arrangements being the same as before detailed.

Russian scale with wires .04 of an inch diameter.
From 0 to 36 inches.

Without Wires.	A Wire under the middle.	Wires under the ends.	Error when curved upwards.	Error when curved downwards.	Sum of Errors.
div.	div.	div.	div.	div.	div.
69	66	75			
71	66	73.5			
71	65	74			
70	67.5	75			
70	63.5	75			
70	64	73.5			
Mean 70.5	65.3	74.3	7.3	1.7	9.0
2.1 Reduction to the Chord.					
72.6					

I now employed wires of .05 of an inch in diameter, and these supported the bar wholly above the surface of the marble.

With wires .05 of an inch diameter.

From 0 to 36 inches.

Without Wires.	A Wire under the middle.	Wires under the ends.	Error when curved upwards.	Error when curved downwards.	Sum of Errors.
div.	div.	div.	div.	div.	div.
38.5	35.5	43.0			
39.5	32.5	39.5			
36.2	31.5	39.0			
Mean 38.1	33.2	40.5	7.0	0.3	7.3
2.1 Reduction to the Chord.					
40.2					

From these last Tables it appears that the sum of the greatest errors to which the Russian scale is liable from flexure is the greatest possible when the versed sine is .04 of an inch, and is then equal to nine divisions of the micrometer, or .00038 of an inch, which, reduced as in the foregoing experiments with the other scales to a versed sine of .01 of an inch, gives for the comparative sum of the errors due to such versed sine .00009, or about one ten-thousandth of an inch.

It is evident that the advantage of this construction consists merely in the employment of a thin plate to receive the divisions, instead of a bar of greater thickness; but a plate sufficiently thin could not well be used without a firm support, which is necessary to protect it from injury, as well as to prevent the plate from being affected by any sudden variation of temperature, either from partial currents of air, or from the vicinity of the person of the observer.

XXIV.—*On the Illumination of Light-houses. By Lieut.* Thomas Drummond, *of the Royal Engineers. Communicated by Lieut. Colonel* Colby *of the Royal Engineers, F.R.S.*

Read June 17th, 1830.

In a former paper which the Royal Society has done me the honour to place among its Transactions*, a method of producing intense light is described, the object of which is to render visible distant stations in geodetical operations. At the conclusion of that paper I ventured to suggest that this method might be found useful for other purposes besides that for which it had been contrived; and more especially for the illumination of light-houses.

Soon after its publication, I received, through my friend Colonel Colby, a communication from the Trinity House, under whose jurisdiction and management the greater number of light-houses on the coast of England and Wales are placed, intimating the intention of that Corporation to give the method proposed a fair trial, whenever the apparatus should be brought to such a state of perfection as would ensure the steady continuance of the light, while at the same time it admitted of being entrusted with safety to the hands of the ordinary attendants. A good deal remained to be accomplished before this degree of simplicity could be considered as attained: nevertheless, the offer of the Corporation was prompt and liberal; and, with the assurance that the time and labour devoted to this object would not be vainly and unprofitably expended, I was anxious to undertake the necessary experiments without delay.

The survey of Ireland, however, had just been commenced; and being employed on that service, I found it impossible to continue these experiments, in the first instance from constant occupation and absence from London, and latterly from a long and severe illness, the consequence of a very laborious and anxious duty in Ireland.

During the last winter, however, I was again able to return to this subject,

* Phil. Trans. 1826, Part III. page 324.

and take part in a course of experiments instituted by order of the Trinity House, and carried on under the direction of the Committee for the Management of Light-houses, with a view to ascertain the relative merits of different methods either adopted or proposed to be adopted in the illumination of light-houses. The result of these experiments I have received permission to communicate to the Royal Society; but in order that the particular methods to which they refer may be better understood and more fully appreciated, a few preliminary observations may be necessary.

The more rude and ancient methods of illuminating light-houses with open coal fires, with common lamps or candles, sometimes aided by reflectors composed of small facettes or plane mirrors*, have in this country been completely superseded even in light-houses of secondary importance; and it may be said that there is only one method now in use for this purpose along the coast of Great Britain and Ireland. This consists in the use of a parabolic reflector of about three or four inches focal length, and from twenty-one to thirty inches in diameter, illuminated by an Argand lamp, seven-eighths of an inch in diameter, placed in the focus. The reflector is hammered out of a plane surface consisting of two plates of silver and copper rolled out together, and though executed with great skill, considering the means, cannot be regarded as a very perfect instrument. This description must be understood as applying only to light-houses under the management of public bodies: with respect to those which have been let to private individuals, I have no very accurate information; but, if they should, on examination, prove to be of an inferior order, it would only be the natural consequence of so pernicious a system.

In fixed lights the number of these reflectors varies according to the portion of the circumference required to be illuminated; but it should not be less than this arc divided by the angle of divergence of the reflected light. At the Eddystone, where the whole circle requires to be illuminated, the number

* The Eddystone till the year 1811 was lighted with 24 wax candles. Up to that time it was in the hands of private individuals; but on the expiration of the lease the Trinity House took it under their own management, and immediately substituted lamps and reflectors. The Bidstone, a leading light to Liverpool, consisted of a large built reflector, about 10 feet in diameter, lighted by an immense spout-lamp with a wick about 12 inches wide, from which a volume of smoke arose that completely intercepted the light from the upper part of the reflector.

should not be less than $\frac{360}{17} = 21°$; if it be less than this, there must be dark spaces diverging from the light-house as a centre, in which nothing but the unassisted light of a single Argand will be visible.

In revolving lights there are five, seven, and even ten reflectors on a side, the number of sides being usually three or four. In the light-houses lately erected on Beachy Head, and on the Perch Rock at the mouth of the river Mersey, there are thirty reflectors in each, disposed on three sides, each bearing ten reflectors. These are the latest, and may be considered as the best specimens of this method of illumination; being about ten times more powerful than the ordinary fixed lights. In some few instances oil gas has been introduced, but the intensity of the flame being very little, if at all, superior to that of an Argand lamp supplied with the best spermaceti oil, little or no advantage can be expected from this introduction, as far as regards the brilliancy of the light, when reflectors are used.

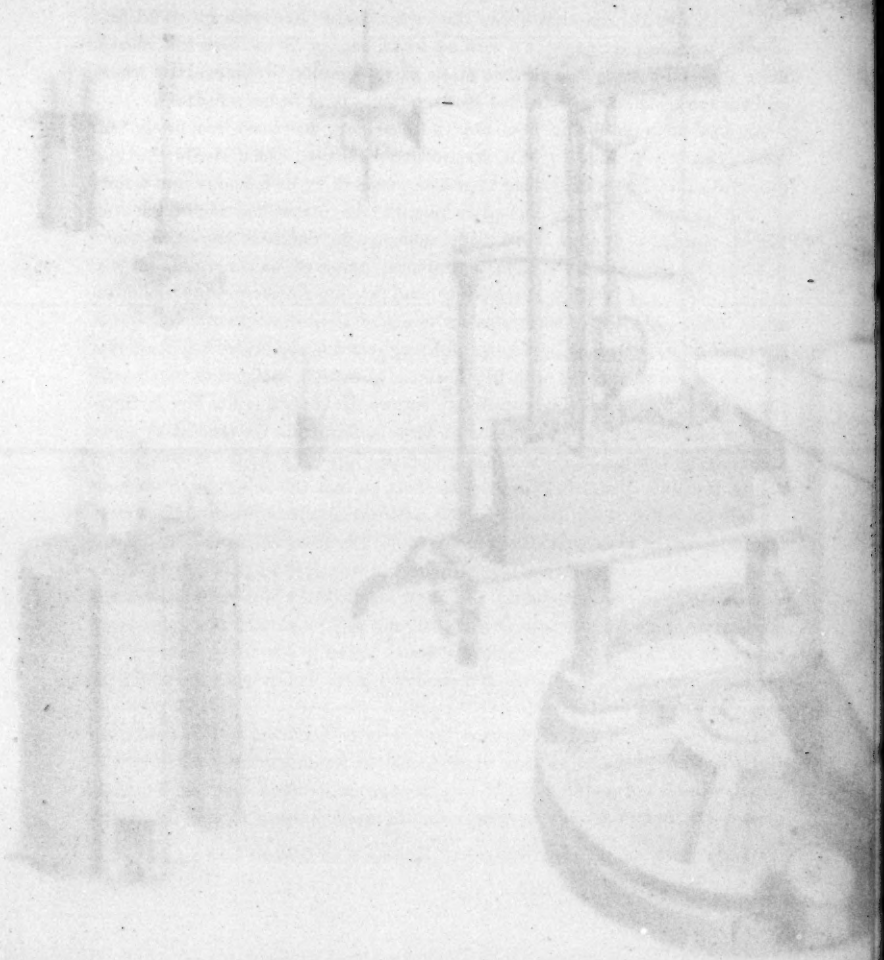
About thirty-eight years ago the experiment was tried, in three or four light-houses, of substituting glass lenses instead of metallic parabolic reflectors. These lenses were 20 inches in diameter, of 19 inches focal length, and about 5 inches thick; but from the imperfection of form and the badness of material, the light transmitted by them appears by our late experiments to be about one-third of that of the reflectors now in use, while their divergence is so small, that at $1\frac{1}{2}°$ on each side of the axis they cease to be visible. With a view probably to remedy these defects, a somewhat extraordinary arrangement was adopted, viz.—the addition of parabolic reflectors behind the lenses. It is true that by this means some addition is made to the direct light of the lens, and, what is of more consequence, the divergent light is increased so that at an angle of about 3° with the axis, it is equal to about thirteen times the light of an Argand. So far therefore the reflector, though but a small portion of it comes into use, contributes to the effect of the lens; but the converse experiment does not appear to have been tried, viz.—how far the reflector was improved by the lens placed before it; otherwise it would quickly have been perceived that the effect of the reflector alone was about double the united effects of the reflector and lens; while at the same time its effective divergence was also greater, being about eight times that of the combined lens and reflector, at an angle of 3° on either side of the axis.

This plan was fortunately never very extensively adopted; and in those light-houses belonging to the Trinity House, where it was tried, it has subsequently been discontinued, and the lenses replaced by reflectors. The North Foreland, however, under the management of the governors of Greenwich Hospital, still remains a solitary example of a method which cannot be too soon abandoned, more especially since the remedy is so easy,—merely to remove the lenses, and leave a free and unobstructed passage to the light of the reflectors.

Another mode, differing from any of those now described, has lately been introduced into France by MM. Arago and Fresnel, which rivals the most powerful of our lights in brilliancy, and surpasses them in economy and facility of management. A large Argand lamp with four concentric wicks, the exterior of which is $3\frac{1}{2}$ inches in diameter, occupies the centre of the light-house. Around this powerful light eight magnificent lenses 30 inches square are disposed, touching each other at the edges, and forming a hollow octagonal prism about the lamp. Above these, smaller lenses of a similar construction, but in the form of trapezoids, are placed, inclining towards the centre till their axes form angles of about 50° with the horizon, at which inclination their sides come into contact, and thus completely inclose the central light. By the intervention of plane mirrors, the beams of light issuing from the secondary lenses are rendered parallel to those of the principal; but by the same means a horizontal deviation of about 7° is given to them, so that this addition to the light is made to contribute to the divergence and consequent duration of light when revolving, rather than to add to its brilliancy. The lens, which is plano-convex, is of a peculiar construction, being formed of separate rings or zones, whose convex surfaces preserve nearly the same curvature as if they constituted portions of one complete lens, the interior and useless part of the glass being removed; so that a section of these zones resembles a wedge placed with the edge uppermost; one side, that next the lamp, being a straight line, the other an arc of a circle*.

The idea of such a lens appears first to have occurred to the celebrated Buffon, when engaged in some experiments on burning-glasses; but he supposed, what is not possible, that it might be ground out of one large piece of glass. Dr. Brewster, in an article on the same subject in the Edinburgh

* Fresnel, Mémoire sur un nouveau système d'eclairage, lu à l'Academie des Sciences, 1822.



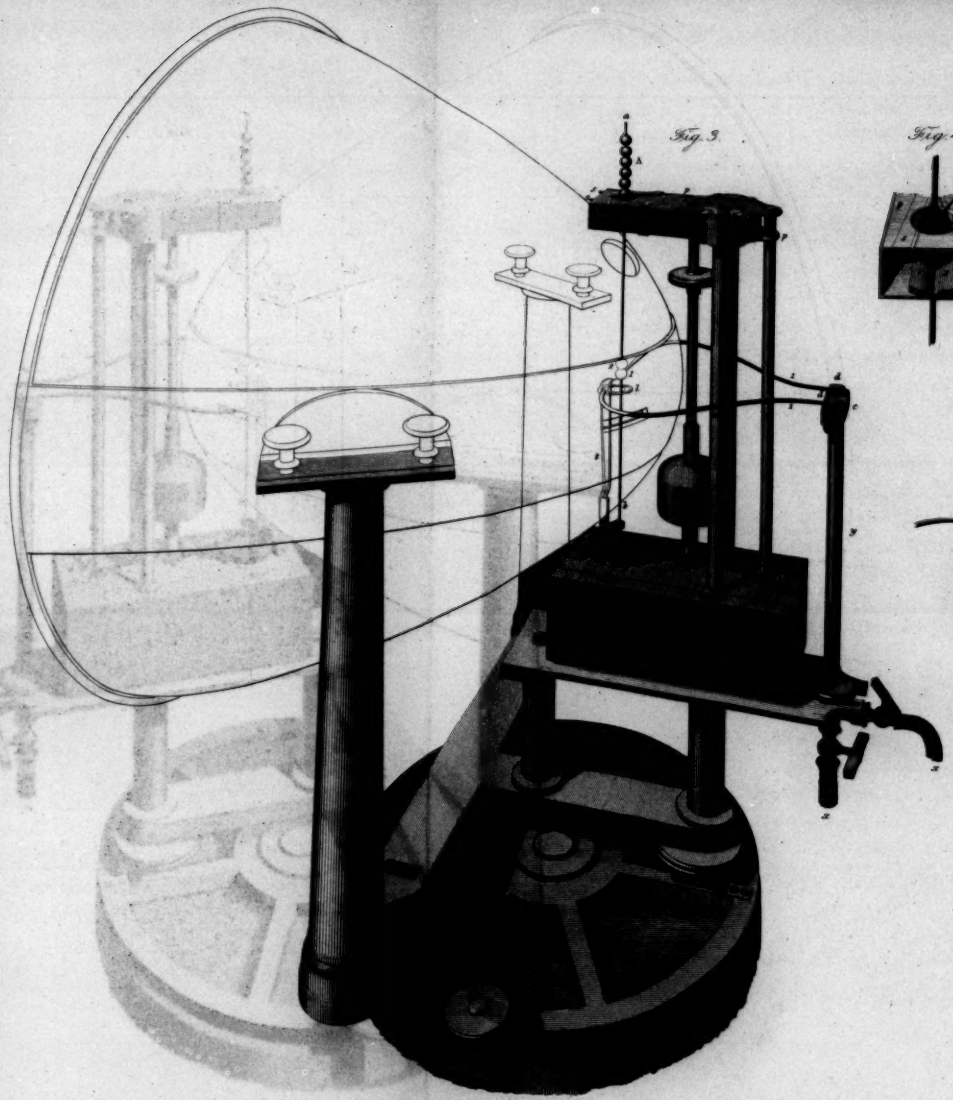


Fig. 3.

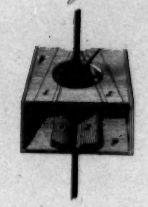


Fig. 4.

Fig. 1.

Fig. 2.

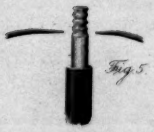


Fig. 5.

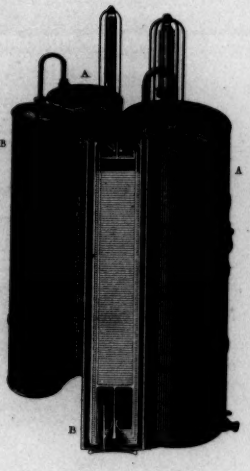


Fig. 6.

Encyclopedia, in 1811, showed that it might be built of separate pieces; and this was an important step, inasmuch as it rendered of easy execution what was before impracticable. To Dr. Brewster therefore the priority of suggesting this improvement is due. To MM. Arago and Fresnel, obviously unacquainted with what had been previously done or recommended, belongs the praise of having first got such a lens constructed, of combining it with a very powerful lamp, and above all of giving it a most useful and beneficial practical application.

The Cordouan light-house at the mouth of the Garonne, the difficult entrance to Bourdeaux, has been fitted up in this manner; and as the lens and lamp used in our experiments were purchased at Paris of the same makers, a pretty accurate estimate may be formed of its merits when compared with the light-houses of this country.

Such are the methods at present in use in the best light-houses of Great Britain or France. The third and last method is that which I have ventured to propose, and in which the light is derived from a source altogether different from the preceding two; a ball or cylinder of lime, intensely ignited, being substituted for the Argand lamps.

For the purposes of a survey, when portability rather than economy is the more important object, this intense heat was obtained by directing a stream of oxygen gas through a flame of spirit of wine: but for a light-house, where, from the long continuance of the light, economy is a primary object, and portability is no longer required, it was desirable if possible to substitute hydrogen gas for the alcohol. As the effects produced in this manner are very remarkable and considerably exceed those formerly obtained, I shall now give a short description of the apparatus, and then proceed to state the results of our experiments.

Plate XII. fig. 1. represents the lamp. The two gases, oxygen and hydrogen, proceeding from separate gasometers, enter at *o* and *h*, but do not mix till they arrive at the small chamber *c*, of which fig. 2. is a section: into this chamber the oxygen gas from the inner tube is projected horizontally through a series of very small apertures, and the hydrogen gas rises vertically through a series of similar apertures at *d*. The united gases then pass through two or three pieces of wire-gauze placed at *e*, and being thus thoroughly mixed, issue through the two jets against the ball *b*. To prevent the wasting of the ball opposite

the two jets, and at the same time to diffuse the heat more equably, it is made to revolve once in a minute, by means of a movement placed underneath the plate *m*, and with which the wire *f*, carrying the ball and passing through the stem, is connected. Notwithstanding, however, this arrangement, the effect of the heat is such as gradually to cut a deep groove in the ball, so that at the end of about 45 minutes it becomes necessary to change it*. In a light-house where it is of essential consequence to maintain a constant light, it would be unsafe to entrust this to an attendant, and hence the necessity of devising some means for remedying this inconvenience. The apparatus represented by fig. 3. is designed for this purpose, and is drawn in the manner in which it is applied to a reflector, the dotted outline of which is shown.

The wire *a b* passes through the focus of the reflector, and upon it are placed the number of balls at A, required for any given time; these, by means of the shears *s*, as shown in fig. 4, are admitted between the plates *p p*, and thence permitted to fall in succession to the focus. No. 1. represents the focal ball; about two minutes before the change, the ball 3. falls into the position 2, where it becomes gradually heated. At the end of that time, the curved support *t*, moving on a pivot, is thrown into the position represented by the dotted line, by the momentary descent of the ring *r*, which, receiving an impulse from the weight *w*, acts upon the extremity *u* of the support. No. 1. falls, but is prevented from descending more than its own diameter by the loop *l*, and No. 2. following it, occupies the focus. The support *t*, being immediately released, returns by the action of a spring to its former position, retains No. 2, and suffers No. 1. to escape through the loop into the cistern.

The wire *a b* and the support *t* revolve together, and carry round the focal ball, which is ignited as in fig. 1. by the two jets *z z*. These jets, which are moveable round the joints *d d*, enter through small apertures cut in the sides of the reflector, and are easily adjusted to the proper distance from the ball.

Wherever the light is required to be diffused equally around, the renewal of the lime may be effected still more easily, by using a cylinder as represented in fig. 5, instead of a ball, which being gradually raised while revolving, brings fresh portions in succession opposite the jets. In a reflector, a cylinder occa-

* When a cylinder is used instead of a ball, a ring of minute crystals is found adhering to the surface above and below this groove.

sions partial shadows at the top and bottom; still, however, the simplicity and certainty with which it may be renewed will probably entitle it to a preference even in this case.

These different instruments were prepared by Mr. Simms of Fleet-street, to whom, for various ingenious suggestions, for the trouble which he has bestowed upon them, and the assistance which he contributed during the progress of the experiments, I am greatly indebted.

The apparatus for supplying the lamps with gas is represented in fig. 6. It consists of two strong cylinders, A, 3 feet high, the one for oxygen, the other for hydrogen: the gas is compressed two or three times in each, the latter by being generated under pressure, the former by being pumped in. To each of these gas-holders, a governor, B, is attached, of one of which a section is shown; by which means, whatever be the variation of pressure in the gas-holder, provided it exceed that of the governor, the gas will issue at x with a uniform and constant stream; in the present instance under a pressure of 30 inches of water. Although this apparatus was of great use in the experiments at the Trinity House, and subsequently at Purfleet, by enabling me to keep within a small compass a supply of gas sufficient for two hours consumption, and even to renew it without impeding the progress of the experiments, yet I may remark, that on a large scale the gasometers required would be much more simple, since compression would no longer be required. This apparatus was made for me by Mr. Samuel Crosley, the ingenious inventor of the gas-governor.

Our first experiments were on the illuminating powers of the different lights, independently of the lenses or reflectors with which they are generally used. The method of shadows and that of equally illuminated surfaces, both depending on the same principle, but requiring different instruments, were employed*; the former after the manner of, and with all the precautions recommended by, Count Rumford †; the latter according to the arrangements proposed by Mr. Ritchie ‡, who was kind enough on this occasion to make several experiments with his own instruments, and without being made acquainted with the results previously obtained by us. The standard we used was an excellent

* Bouguer, Traité d'Optique.

† Phil. Trans. 1794, Part I. page 67.

‡ Phil. Trans. 1825, Part I.

Argand lamp $\frac{7}{8}$ inch in diameter, supplied with the finest spermaceti oil, and capable of supporting a flame $1\frac{3}{4}$ inch in height. The following results were obtained:

			Standard.	Wax candles, $1\frac{1}{2}$ inch diam., the wick consisting of 40 threads.
French lamp, Mean of	5	observations by shadows . .	1	= 9.1
———	9	——— . .	8.7 } 10.4*	= 94.6
———	6	——— by illuminated surfaces	12.1 }	
Oil gas burner, same diam. as standard	4	——— by shadows	0.95	= 8.7
Lime-ball or cylinder $\frac{3}{8}$ inch diam.	2	———	9.2	
	2	———	10.4	
	4	——— by illuminated surfaces	18.6 } 13.5	= 122.9
	10	———	16.0	

The light of the ball, depending upon the intensity of the heat, is very different at different parts, being greatest opposite the jets, and diminishing towards the sides. The mean of the greatest and least intensity is taken in the above Table; and moreover, though the greater number of observations might appear to warrant giving greater weight to their results, yet being made on the same day, and under the same circumstances, it was found that the results seldom differed, whatever might be the number of observations; hence the arithmetical mean is taken, and we obtain this remarkable result,—that the light emitted by a lime-ball only $\frac{3}{8}$ of an inch in diameter, heated by two jets, is equal to 13 Argand lamps.

With respect to the intensity or intrinsic brightness of the different lights, the property on which their utility in light-houses more immediately depends, we have the following results:

The intensity of				Mean.	
French lamp, Mean of	11	observations by shadows . .	= 4.1 }	4.0	times the intensity of the Standard.
———	8	——— . .	= 3.8 }		
Oil gas	6	——— . .	= 0.85 †		
Lime	6	——— . .	= 263.9 }	264.1	
———	3	——— by illuminated surfaces	= 264.4 }		

* The result given by FRESNEL, in the memoir quoted above, considerably exceeds this, being stated at 17 lamps of Carcel. I know not to what cause this difference is to be attributed.

† This low degree of intensity indicates impurity in the gas.

These results were obtained by screening the different lights, and then placing equal apertures opposite each, changing the apertures and taking the mean to destroy the effect of any inaccuracy in size. The intensity of the lime-ball being therefore 264 times that of the Argand lamp, a single reflector illuminated by the former will be equal to 264 reflectors illuminated by the latter; but the divergence of the reflected light, depending upon the size of the luminous body in the focus, will be smaller with the ball than with the lamp in the proportion of about 3 to 8: hence in such a light-house as that of Beachy Head, 8 reflectors may be substituted for 30, and yet an effect would be produced 26 times greater than that of the present light, the most perfect of its kind in this country.

By similar experiments it was found that the French lens was equal to 9.1 reflectors; and if the effect of the additional lenses and reflectors which ought to accompany it, and which has been estimated at one-seventh, be added, then the lens is equal to 10.4 reflectors. In like manner, therefore, the effect of a single reflector with a lime-ball would be equal to 25 times that of such a combination of lenses.

Such appear to be the singular and important results of our late experiments at the Trinity House, made as they have been with every precaution by different individuals, with different instruments, and unbiassed by the knowledge of each other's results. I see no reason to doubt their accuracy; and the comparative appearances of these different lights, when exhibited at a distance of ten miles, to which I shall presently allude, though not admitting of being reduced to numbers, confirm the striking superiority of this method of illumination.

It may now perhaps be asked, At what expense can such a light be maintained? Can the gases by which the requisite heat is produced be procured at such a price as to compete with oil or coal gas? The data I possess for forming an estimate of the expense of the gases are very scanty, but the quantity consumed was accurately determined; at the same time the consumption of the other lights was also tried, and the results are as follow:

	Consumption in 3½ hours.	Expense per hour.
An Argand lamp seven-eighths of an inch in diameter	1 gill	0.69 penny
The same placed in a reflector . .	1⅓ gill . . .	0.83 penny
The French lamp	2 qts. ⅓ pt. . . .	1*s.* 2½*d.*

The lime requires 4 cubic feet of hydrogen and 2 of oxygen per hour, and the probable expense is 5*d.* per hour.

In a revolving light of the first class, containing 30 reflectors, the expense per hour would therefore be about 2*s.* 1*d.* If the French method were employed, the increase of light would be $\frac{1}{20}$th, and the expense only 1*s.* 2$\frac{1}{2}$*d.* per hour. If six reflectors illuminated with lime-balls were used, which would probably be sufficient, the probable expense would be 2*s.* 6*d.* per hour, and the increase of light 26 times.

If this estimate be erroneous, I think it will prove to be so in excess: admitting, however, that the expense should, in the first instance, somewhat exceed what has been stated, it may in this as in every similar instance be expected that after a little experience a considerable reduction would be effected. This is a new source of artificial light, differing from every other at present in use, and the materials by which it is produced are among the most abundant products of nature; but never having yet been applied on a great scale to any practical purpose, it has not hitherto been an object to obtain them in a separate state at a small expense. When this is effected, it will no doubt receive many useful and important practical applications.

Meanwhile, however, the case in question may perhaps be regarded as one where expense ought not to be a primary object of consideration. On all ordinary occasions, the preference of one mode of illumination to another is a question of convenience, luxury or economy; but in this it assumes a more important character, for it involves to a great extent the preservation of life and property.

To complete the preceding account, it only remains to add a description of the appearances presented by the different lights when exhibited at a distance; and to those who have entered with any degree of interest into the above details, such a description, it is hoped, cannot fail of proving acceptable.

The experiments at the Trinity House being concluded, the whole of the apparatus was removed to Purfleet, where on a knoll of chalk about 100 feet above the river a temporary light-house had been erected, and being fitted with the requisite machinery, the different lights were made to revolve in succession, and the appearance which they presented, as well as the duration of the light, were observed from the Trinity Wharf at Blackwall, a distance in a straight line of 10$\frac{1}{4}$ miles.

The four faces of the revolving machine were thus occupied:

No. 1. A single reflector 21 inches diameter, 3 inches focal distance, with an Argand lamp.

No. 2. Seven reflectors with ditto.

No. 3. French lens, with its lamp.

No. 4. Single reflector with lime-ball.

The respective lights were accurately placed in focus.

On the evening of the 10th of May, the machine performing one revolution in eight minutes, Captain PELLY of the Trinity House made the following observations on the different lights from the Trinity House Wharf, Blackwall, $10\frac{1}{4}$ miles distant.

No.	Duration.		Divergence.	Computed maximum Divergence.
	min.	sec.	°	°
1	0	25	17.40	17.81
2	0	25	17.40	17.81
3	0	7	5.17	5.18
4	0	9	6.12	6.7

When No. 4, the reflector lighted with the lime-ball, was turned towards the Wharf, the light was so great that the shadow of the hand and fingers was distinctly visible even on a dark brick wall, while no such effect was discernible when the other lights were turned in the same direction.

In order more justly to estimate their comparative effects, No. 4. was removed to a temporary tent about twenty-five yards to the right of the light-house, as far as the edge of the cliff would permit, and on the evenings of the 25th and 31st May regular series of experiments were made. Being engaged at Purfleet, directing these exhibitions, I never had an opportunity of witnessing their effects at Blackwall; but Captain BASIL HALL, R.N., who from the interest which he took in these experiments was an attentive observer of all that occurred, has at my request kindly favoured me with the following interesting account:

4, St. James's Place, 1st June 1830.

"My dear Sir,

"You wished me to take particular notice of last night's experiments with the different kinds of lights exhibited at Purfleet, and observed at the Trinity

Wharf, Blackwall; but I have little to add to what I told you respecting those on the evening of the 25th instant: indeed it is not within the compass of language to describe accurately the details of such experiments, for it is by ocular evidence alone that their merits can be understood.

"Essentially the experiments of last evening were the same as those of the 25th, and their effects likewise. The degrees of darkness in the evenings however were so different, that some particular results were not the same. The moon last night, being nine or ten days old, lighted up the clouds so much, that even when the moon herself was hid, there was light enough to overpower any shed upon the spot where we stood by your distant illumination: whereas on the 25th, when the night was much darker, the light cast from the temporary light-house at Purfleet, in which your apparatus was fixed, was so great that a distinct shadow was thrown upon the wall by any object interposed. Not the slightest trace of any such shadow, however, could be perceived when your light was extinguished, and any of the other lights were exposed in its place.

"In like manner on the evening of the 25th it was remarked by all the party at the Trinity Wharf, that, in whatever direction your light was turned, an immense coma, or tail of rays, similar to that produced by a beam of sun-light in a dusty room, but extending several miles in length, was seen to stream off from the spot where we knew the light to be placed, although, owing to the reflector being turned too much on one side, the light itself was not visible.

"Now, last night there was none of this singular appearance visible; but whether this was caused by the presence of the moonlight, or by the absence of the haze and drizzling rain which fell during the evening of the 25th, I cannot say. I had hoped that the appearance alluded to was to prove a constant accompaniment to your light, in which case it might, perhaps, have been turned to account for the purposes of light-houses. If in hazy or foggy weather this curious effect of reflected light from the atmosphere be constant, it may help to point out the position of light-houses, even when the distance of the observer is so great that the curvature of the earth shall render it impossible for him to see the light itself.

"The following experiments tried last night were the same as those of the 25th, and certainly no comparative trials could be more fairly arranged.

"Exp. I. The first light exposed was the single Argand burner with a reflector. This was quite distinctly seen, and all the party admitted it to be a good light. After several minutes this was put out.

"Exp. II. The seven Argand burners were next shown, each in its reflector; and this was manifestly superior to the first; but how much so I cannot say, perhaps four times as conspicuous. Both these lights had an obvious tinge of brown or orange.

"Exp. III. The third light which was exposed, (on the seven Argands being put out,) was that behind the French lens; and I think it was generally admitted by the party present, that this light was whiter and more intense than that from the seven Argands, though the size appeared very much the same.

"Exp. IV. The fourth light was that which you have devised, and which, instead of the clumsy word 'Lime', ought to bear the name of its discoverer. The Drummond light, then, the instant it was uncovered, elicited a sort of shout of admiration from the whole party, as being something much more brilliant than we had looked for. The light was not only more vivid and conspicuous, but was peculiarly remarkable from its exquisite whiteness. Indeed there seems no great presumption in comparing its splendour to that of the sun; for I am not sure that the eye would be able to look at a disk of such light, if its diameter were made to subtend half a degree.

"The next series of experiments was the most interesting and decisive of all. Each of the lights above enumerated, viz. the single Argand burner, the seven Argands, and the French lens, were exposed, one at a time, in company with your light, in order to try their relative brilliancy.

"First comparative Experiment.—The single Argand burner was first exposed to this comparative ordeal, and nothing could be more pitiable than the figure it cut. Many of the party could not see the Argand light at all; while others could just detect it 'away in a corner,' as some one described it. It was also of a dusky orange tinge, while your light was of the most intense whiteness*.

"Second comparative Experiment.—The seven Argand burners were now substituted in place of the single light. All the party could now see both

* To many, the rays from the brighter light appeared, when seen with the naked eye, to extend across and envelope the fainter light, though the perpendicular distance between them was twenty-five yards.

lights, but the superiority was not much less obvious. I really cannot affix a proportion either as to size or brilliancy; but I should not hesitate to say that your light was at least six or eight times as conspicuous; while in brilliancy, or purity, or intensity of light, (for I know not precisely what word to use to describe the extreme whiteness,) the superiority was even more remarkable. All this which I have been describing was expressed, and appeared to be quite as strongly felt by the rest of the company, to the number, I should suppose, of five-and-twenty or thirty persons, who were all closely on the watch.

"Third comparative Experiment.—The next comparative trial was between the French lens and your light. The superiority here was equally undeniable; though the difference in the degree of whiteness was not so remarkable. The French light, however, is so nearly similar to that from the seven Argands, that the comparison of each of them with your light gave nearly the same results, and all equally satisfactory on the score of your discovery.

"Final Experiment.—The flashes with which the experiments concluded were very striking, and might I think be turned to great account in rendering light-houses distinct from one another. The revolutions were not effective, and, as I said before, there was no appearance last night of those enormous comets' tails which swept the horizon on the night of the 25th, to the wonder of all who beheld them: neither could there be detected the slightest trace of any shadow from the light thrown towards us, and I suspect none will ever be seen, when the moon, whether the night be clouded or not, is of so great a magnitude.

"Such is the best account I can give of what we witnessed; and I need only add that there seemed to be amongst the company but one opinion of the immense superiority of your light over all the others brought into comparison with it.

"I am, &c.

"Basil Hall."

The advantage of such a light being fully recognised, attention may now be exclusively directed to remove some of those minor obstacles that might render its use in light-houses objectionable; and I have great pleasure in adding that the Trinity Corporation are desirous that every facility in their power should

be afforded with a view to effect this object, and that a series of preliminary experiments is accordingly to be carried on at their expense.

Simple as the apparatus and the experiments now described may appear, they have occasioned more trouble and anxiety than would be supposed by those who have not been engaged in similar pursuits; but, on the other hand, I fully acknowledge the encouragement derived from the interest which they appeared to excite.

His Royal Highness the Duke of CLARENCE, Master of the Trinity Corporation, was pleased to be present on one occasion, and remained more than an hour, entering with great interest into the details, and expressing himself much gratified with the effects which were produced.

Sir GEORGE COCKBURN and Mr. BARROW from the Admiralty, and several other naval men whose opinions on such subjects are entitled to the utmost deference, not only attended at the Trinity House, but went afterwards to Blackwall to observe the relative appearance of the lights when exhibited at a distance. The night of the 31st May had been appointed by the Deputy Master for this purpose; and, being desirous that the subject should be fully scrutinized, I was glad to learn on returning from Purfleet that, besides the gentlemen immediately connected with the Trinity House, the experiments described in Captain HALL's letter had been witnessed by Admiral Sir THOMAS HARDY; by the LORD ADVOCATE of Scotland, one of the Commissioners for the northern light-houses; by Sir THOMAS BRISBANE, Colonel COLBY, Captain BEAUFORT, Hydrographer to the Admiralty, and several other individuals eminent for their professional and scientific attainments; and as far as I have been able to learn, the opinions of these distinguished persons coincide with those expressed by Captain HALL, who in fact, as he has given me authority to state, endeavoured to frame his account of what passed in strict conformity with the general sentiments of the party, and neither to exaggerate nor under-rate any of the results.

It now only remains for me to perform the agreeable task of bearing testimony to the liberal spirit evinced by the Trinity Corporation on this occasion, and to the desire which they have manifested of facilitating, by every means in their power, the introduction of this method of illumination into light-houses. Indeed I hesitate not to express my belief that, if this do not take place, it will arise

from some insurmountable difficulties in the way of its practical application, and not from the want of a full and impartial trial on the part of that body, to whom these establishments are entrusted.

I owe this acknowledgement in an especial manner to JOHN WOOLMORE, Esq. the Deputy Master of the Corporation, to Captains CLARKE, PELLY and BROWNE, the gentlemen constituting the Light Committee, for the fairness and impartiality of their decisions, as well as the indulgence which they extended to those defects inseparable from a new apparatus; and to Mr. HERBERT the Secretary, for his uniform desire to promote every arrangement that appeared likely to bring this inquiry to a satisfactory termination.

To these acknowledgements I may be permitted to add my obligations to my commanding officer and friend Colonel COLBY, for the facilities he has afforded me in carrying on these experiments, and for the advice and assistance which he has on this and many other occasions so kindly rendered me.

London, June 17th, 1830.

XXV. *On the electro-magnetic properties of metalliferous veins in the mines of Cornwall.* *By* Robert Were Fox *of Falmouth. Communicated by the President.*

Read June 10th, 1830.

In one of my communications to the Cornwall Geological Society on the high temperature of the interior of the earth, I ventured to express a belief that mineral veins, and the internal heat, are connected with electrical action. This opinion, founded as it was on the curious arrangement of the veins, &c. in primitive rocks, I have had the satisfaction to find confirmed by experiments made in some of the mines of Cornwall; and, I doubt not that the existence of electricity in metalliferous veins similarly circumstanced, and capable of conducting it, will prove to be as universal a fact, as the progressive increase of temperature under the earth's surface is now admitted to be, much as my conclusions on this point were at one time controverted.

In my first experiment I did not succeed in detecting any electricity; but in my second I had the gratification to observe considerable electrical action.

My apparatus consisted of small plates of sheet copper, which were fixed in contact with ore in the veins by copper nails, or pressed closely against it by wooden props, stretched across the "levels" or galleries. Between two of these plates at different stations, and a galvanometer*, a communication was made by means of copper wire one-twentieth of an inch in diameter, which was at first coated with sealing-wax; but afterwards this precaution was dispensed with.

* It may be proper to describe the galvanometer employed in making many of my electro-magnetic experiments. The magnetic needle was three inches and a quarter long, one-eighth of an inch wide, and one-twenty-eighth thick. It was inclosed in a box four inches square and one inch in depth, having a plated copper wire one-fiftieth of an inch in diameter coiled round it twenty-five times. No magnet was used to neutralize the terrestrial polarity. In my earlier experiments a less delicate galvanometer was employed.

The accompanying sketches are intended to represent the plans adopted, and the results obtained in various mines, and under different circumstances. In some instances nearly 300 fathoms of copper wire were employed*.

The intensity of the electro-magnetic action differed greatly in different places:—in some cases the deviation of the needle was inconsiderable, in others it went completely round the circle. In general it was greater, cæteris paribus, in proportion to the greater abundance of copper ore in the veins, and in some degree perhaps to the depth of the stations;—and where there was little or no ore, there was little or no action. Hence it seems likely that electro-magnetism may become useful to the practical miner in determining with some degree of probability at least, the relative quantity of ore in veins, and the directions in which it most abounds.

When the distance of the plates from each other in a horizontal direction was only a few fathoms, and the copper ore between them was plentiful, and uninterrupted by non-conducting substances, or the workings in the mine, no action occurred, owing no doubt to the good conducting power of the vein; but where a cross vein of quartz or clay happened to be between the plates under similar circumstances, the action was usually great.

When the communication was established between two plates at different depths on the same vein, or between different veins, whether at the same level or otherwise, the electrical action was in general the most decisive. In fact, veins which in some instances were almost destitute of ore, and did not affect the needle per se, did so, though perhaps only in a slight degree, when electrical communications were made between them.

It will be seen that the direction of the positive electricity was in some cases from east to west, and in others from west to east; and when parallel veins were compared, its general tendency was, I think, from north to south, though in several instances it was the reverse. In veins having an underlie towards

* I am indebted to the kindness of my brother, Lewis Fox, for the experiments referred to in Fig. 8, 13, & 22; to W. J. Henwood for those in Fig. 8, 25, & 26, and to Richard Tregaskis for those represented in Fig. 16, 17, & 27. They moreover all assisted me materially on several occasions in making my experiments, especially my brother, who was my companion in most of my descents into the mines. To the various mine agents I am also under particular obligations, for their great attentions and personal assistance; some of them having remained with us underground ten or eleven hours together.

the north, the east was commonly positive with respect to the west; but in veins dipping towards the south, the contrary was observed, with one exception only, and that under rather unusual circumstances. (See Fig. 27.) In comparing the relative states of veins at different depths, the lower stations appeared to be negative to the upper; but exceptions sometimes occurred when a cross vein of quartz or clay intervened between the plates, and the higher one was on the negative side with respect to the horizontal currents. (See Fig. 8. & 12.)

In such cases it may be supposed that there is an accumulation of electricity in different states, on the opposite sides of the non-conducting vein. Such intersections of ore veins, and their being often very rich to a great depth in one direction and not in another, added to their varying underlie at different depths, which is not unfrequently reversed, may tend to produce apparent anomalies in experiments of this nature.

At Huel Jewel mine, I obtained results between a heap of copper ore at the surface, and a plate fixed at different depths against the ore in the vein; the latter becoming more negative, in proportion to the depth at which it was placed. Piles of copper ore at the surface did not act on the needle when tried together, independently of veins, nor was it to be anticipated that they would.

It is not improbable that the progressive increase of negative electricity observed in descending into our mines, if hereafter confirmed, may be found to be connected with the progressive increase of temperature. I have not, however, discovered any distinct connection between them at the same level, but then the differences of temperature are comparatively small. Nor does the electricity appear to be influenced by the presence of the workmen and candles, or by the explosion of gunpowder, although some veins of copper ore were blasted on different occasions in the immediate vicinity of the copper plates. And at a very productive copper vein in Great St. George Mine, the ground is so soft that gunpowder is not used; yet the needle was powerfully acted upon by the electricity it contained. On this occasion, as well as on some others, I remained with the galvanometer at the surface, letting the wires down through the shafts; and in this manner I have sometimes found the electricity act with

considerable energy, so as even to cause the needle to revolve with some velocity.

In connection with the electricity of veins, I deemed it desirable to ascertain the relative power of conducting galvanic electricity possessed by many of the metalliferous minerals; and it appeared to be in about the following order, viz.

Conductors.

Copper nickel,
Purple copper,
Yellow sulphuret of ditto,
Vitreous ditto,
Sulphuret of iron,
Arsenical pyrites,
Sulphuret of lead,
Arsenical cobalt,
Crystallized black oxide of manganese,
Tennantite,
Fahlerz.

Very imperfect conductors.

Sulphuret of molybdenum,
Sulphuret of tin, or rather bell-metal ore.

Non-conductors.

Sulphuret of silver,
Ditto of mercury,
Ditto of antimony,
Ditto of bismuth,
Cupriferous ditto,
Realgar,
Sulphuret of manganese,
Ditto of zinc,
Mineral combinations of metals with oxygen, and with acids.

All the conductors of galvanic electricity were so likewise of common electricity; to which may be added the oxide of tin, and, in a less degree, the sulphurets of bismuth and silver, the phosphate of manganese, and a few of the oxides. Sulphuret of zinc appeared to be a more perfect non-conductor of common electricity as well as the sulphuret of antimony, than the red oxides of those metals.

Amongst the rocks prevalent in Cornwall, clay-slate or "killas" seemed to possess the property of conducting common electricity in a slight degree, but only in the direction of its cleavage, perhaps owing to the moisture it retained.

I mention these facts in some detail, because it is curious to observe that the conducting power of metallic ores appears to have no reference to any of the electrical or other properties of the metals in a pure state, or to the proportion of them in combination. Silver and mercury, for example, are combined with, comparatively, very small quantities of sulphur;—and zinc, which seems to hold an opposite place to silver in the electrical scale, is also found in combination with a much less proportion of sulphur than is contained in copper pyrites, though the latter is one of the best mineral conductors of electricity.

There are many other analogous examples, which prove that no conclusion can be drawn à priori, from the nature or chemical arrangements of minerals, as to their relative electrical properties.

Much time and attention have been bestowed by geologists on the consideration of the origin and comparative ages of veins, and but little, I apprehend, on the purposes for which they are designed.

It appears to me that it will prove a vain attempt to reconcile a multitude of facts observable in our mines with any known natural causes.

I may refer to a few of them:—

1st. The very oblique descent of a large proportion of the veins into the earth, in some cases in very hard rock, and in others in ground so soft that it would immediately fall in, however small the excavation, without being completely supported by timber. Were it possible to conceive fissures to exist under such circumstances, it is not reasonable to suppose that they would not take the direction in which the resistance would be least, that is, either the vertical, or the line of the cleavage of the rocks.

2nd. Veins are often divided into branches, which unite again at a consider-

able depth, including between them vast portions of rock perfectly insulated by the ore or vein-stones from the general mass: these, it is evident, could not have existed as fissures for a moment.

3rd. Veins are continually subject to changes in their horizontal direction and underlie; their size also often varies exceedingly, one part being many times wider than another, without any reference to their relative position or depth under the surface.

4th. Although a portion of their vein-stones are usually quite distinct in their characters from the rocks they traverse, they are generally, in part, of the same nature, and vary with the containing rocks, whether granite, elvan, killas, &c.; and they are commonly too regularly arranged in the veins, and are found inclosing insulated portions of the ore, &c. in their very substance, to admit of the idea of their having been originally mere broken fragments of the inclosing rocks.

At Dolcoath Mine there is an instance of one ore vein intersecting another at different depths, and being itself intersected and even shifted by the same vein at a greater depth. I have given a sketch of these intersections in Fig. 27, (for which I am indebted to Captain Petherick of that mine,) as I am not aware that it has been before noticed.

Many other facts might, if it were necessary, be accumulated, relative to the position and intersections of veins, as well as the nature and arrangement of their contents, which, with those I have stated, are calculated to throw entire discredit on the various hypotheses which have been invented to account for their origin. But my object is, rather to suggest whether the arrangement of veins, &c. does not argue design, and a probable connection with other phænomena of our globe.

Metalliferous veins, and those of quartz, &c. appear to be channels for the circulation of the subterraneous water and vapour; and the innumerable clay veins or "flucan courses" (as they are termed in Cornwall,) which intersect them, and are often found contained in them, being generally impervious to water, prevent their draining the surface of the higher grounds as they otherwise would, and also facilitate the working of mines to a much greater depth than would be practicable without them.

With respect to their electrical properties, it may be observed, that ores

which conduct electricity have generally, in this county at least, non-conducting substances interposed in the veins between the ore and the surface. Thus a brown iron ochre with quartz, &c. named "gossan" by the miners, is almost invariably found resting on copper. Sulphuret of zinc occurs sometimes in the same situation, both with regard to copper and lead; but tin ore, which is a non-conductor, is without either, and is mostly found nearer the surface than copper.

Tin veins are usually intersected by those of copper when they do not coincide in their horizontal direction or underlie; thus, in this case, the conducting veins traverse the non-conducting ones. And when two veins of copper meet at opposite angles in descending, they are, I apprehend, generally found to be unproductive at and near the place of junction; but when they unite, proceeding downward in the same direction but at different angles, they are commonly observed to be enriched. These facts appear curious when regarded in connection with the opposite currents of electricity in veins having opposite dips.

There are some districts in this county in which the ore veins have generally a north underlie, and in others the south prevails; and it often happens that when lodes occur which deviate from the prevalent underlie of the others, in any district, the former are intersected, and sometimes shifted by the latter. This is strikingly the case in numerous mines in the parishes of St. Agnes and Perran.

The usual horizontal bearing of the copper and tin veins in our principal mining districts, appears to be nearly E. and W., or rather from E.N.E. to W.S.W.; but in others they deviate materially from these directions, sometimes to E.S.E. and W.N.W.: indeed, in some places this is the prevailing course of the veins of ore.

When veins containing the sulphuret of silver occur, (which as I have before stated is a non-conductor of electricity,) they are generally found nearly at right angles to the copper and tin veins, and seem thus to assume in great measure the character of cross veins of quartz, clay, &c.

With respect to the two latter, it has been observed that when they shift the ore veins, there is frequently to be found in them scattered stones of ore, or a small vein of it, or "leader" (to use a mining term), between the dislocated parts of the lode. This is also the case often with slides; so that, although

the horizontal transfer of the electricity may be much impeded, it does not seem to be wholly intercepted. The quartz contained in cross veins is usually of a fibrous or radiated texture, and differs materially from that found in the east and west veins.

All our mining districts abound more or less with veins or dykes of a rock generally possessing a porphyritic character, termed by the miners "Elvan courses." Their width is extremely various, sometimes as much as fifty fathoms and upwards. Their direction in general is nearly N.E. or E.N.E. to S.W. or W.S.W., and their underlie is with few exceptions towards the N.W., and at various angles from the perpendicular, often exceeding 45°. They are penetrated by ore-veins in almost every direction, from their greater underlie, and usually more considerable deviation from an east and west bearing than the latter. It has been observed that copper and tin lodes generally become changed in quality whilst in the elvan*; and indeed this remark applies to any change of rock: thus a vein productive in granite commonly becomes barren in killas, and vice versâ.

Many of the phænomena above referred to, bear striking analogies to common galvanic combinations, and the discovery of electricity in veins seems to complete the resemblance.

I have been informed by intelligent persons who have visited some of the mining districts of Mexico, Guatimala, and Chili, that there is a general resemblance between the veins, elvan courses, &c. in some parts of those countries and our own; and I think it has been noticed by Baron Humboldt, that the stratification of primitive rocks in different, and far distant parts of the world, has a general tendency from the N.E. towards the S.W.

Such analogies become highly interesting when regarded in connection with terrestrial electricity, magnetism, and heat; for if it be granted that the two latter increase in intensity at great depths in the earth, they are evidently so connected with electrical action that the augmentation of it also, in the interior of the globe, may be reasonably inferred.

However this may be, assuming that metalliferous veins exist more or less in primitive rocks generally, (and experience favours this assumption, whether we

* It has been remarked of copper lodes, that they are often improved in quality whilst in elvan, particularly if it be not very hard.

refer to the new mines which have been discovered in various parts of North and South America, Siberia, Ireland, &c. or to the mining county of Cornwall, in which whole districts have comparatively of late been found abounding with mineral treasure, where none had been formerly suspected to exist,) it may I think be presumed, that the electrical currents, which so affect the needle in the galvanometer, may likewise influence the direction of the magnetic needle on the surface of the earth: at least no explanation of this phenomenon appears to be so plausible, or so well connected with ascertained facts. Even the cause of the variations of the needle, mysterious as it has hitherto appeared to be, may probably be referred to the relative energies of the opposing electrical currents, which are perhaps subject to occasional modifications; and the appearance of earthquakes and volcanic action, from time to time, seems to countenance the probability of such changes.

Nor should it be overlooked in reference to this view of the subject, that the oblique bearing which is generally observable in the strata and veins, with respect to the equator, causes them, as it were, to cross at opposite sides of the globe in the same parallels of latitude, so that their tendency, if any, must necessarily be to produce more than one magnetic pole in each hemisphere. Thus, in this respect also, the hypothesis accords with the interesting fact lately announced;—of Professor Hansteen having ascertained the existence of a second magnetic pole within the arctic circle. The revolution of the earth on its axis from west to east seems moreover to harmonize with the idea of oblique electrical currents; since rotation in the same direction may be produced by corresponding electro-magnetic arrangements.

Before I conclude, I will briefly mention a few facts relative to the temperature of some of the mines in Cornwall.

At Tingtang copper mine, in the parish of Gwennap, at the bottom of the engine shaft, which is in killas, and 178 fathoms deep, the water about two months ago was at the temperature of 82°. In 1820, when the same shaft was 105 fathoms deep, the temperature of the water was 68°: thus an increase of 14° has been observed in sinking 73 fathoms, which is equal to 1° in 5 fathoms.

At Huel Vor tin mine, near Helston, the water was 69° at the bottom of a shaft 139 fathoms deep, in the year 1819. It is now 209 fathoms deep, and

the temperature is 79°, which gives a mean increase of 1° in sinking 7 fathoms. This part of the mine is in killas.

The highest temperature of the water at the bottom of Poldice copper and tin mine in the parish of Gwennap, in 1820, which was then 144 fathoms under the surface, was 80°. It is now 176 fathoms deep, and the temperature is 99°; and in a cross level 20 fathoms further north, the water is 100°*.

The two last-mentioned temperatures are the highest hitherto observed in any of the mines of this county; and the increase is equal to 19° in one case, and 20° in another, in sinking 32 fathoms, or 1° for 1½ fathom. Three persons only were employed at a time near each of these stations, and the water pumped up from this part of the mine was estimated at 1,800,000 gallons in twenty-four hours; and I found on examination that it contained a considerable quantity of common salt in solution.

* The thermometer used on these occasions was compared with others, and corrected one degree.

Explanations.

Sections of Cross Veins marked thus

Section dotted, Granite.

Short line section, Clay Slate.

The arrows mark the direction of the positive electricity through the wire, which is represented by the black lines, and the number of cross marks on the arrows are increased according to the increase of electrical intensity.

⊕ Marks the stations of the Galvanometer.

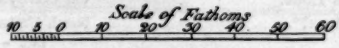
⊃ The copper plate at the bottom of a level.

⊃ Ditto at the back of a level. They are shaded in proportion to the richness of the veins, which are all of copper ore, except Fig. 11 & 18, and those otherwise marked in Fig. 19 & 21.

B.S. Bottom of the Shaft.

fm D. Depth from Surface in fathoms.

00° Temperature at the Stations of the plates.



Wires corresponding with the directions of the ore Veins or Lodes, from Fig. 1. to 18.

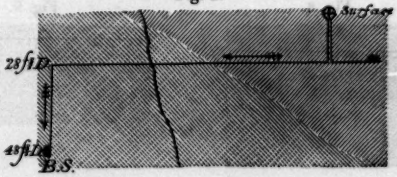
Huel Jewel.

South Lode.

North Underlie 23°.

Fig. 1.

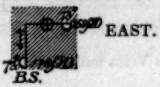
Fig. 2.



Tingtang Mine.

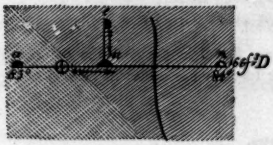
On Roche's Lode.
North Underlie 12°.
Fig. 3.

On junction of Old and Middle Lodes.
North Underlie 24°.
Fig. 4.



Poldice.

Old Ore Lode.
North Underlie 23°.
Fig. 5.



positive. negative.
a with *n*
a — *m* strong action.
e — *m* ditto

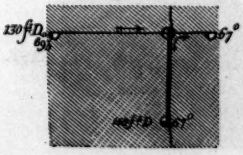
Huel Unity.

On William's Lode.
North Underlie 27°.
Fig. 6.

On Garby's Lode.
North Underlie 27°.
Fig. 7.

Huel Damsel.

On South Lode.
North Underlie 20°.
Fig. 8.



Dolcoath Mine.

On Richard's Lode, perpendicular.
Fig. 9.

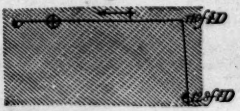
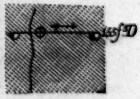


Fig. 10.

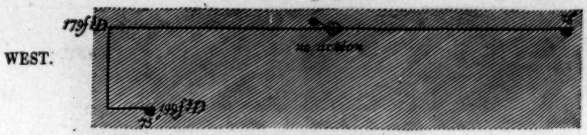


Huel Vor.

On Tin Lode.

North Underlie 33°.

Fig. 11.

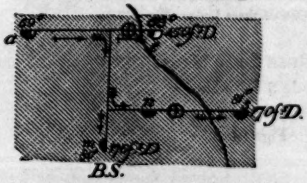


Tresavean Mine.

On Main Lode.

South Underlie 5° to 10°.

Fig. 12.

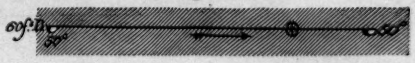


positive. negative.
a with e
a — m
a — n
a — s strong action.
n — s ditto
e — s
m — s
m — e very slight.
n — e ditto

North Lode.

South Underlie 3°.

Fig. 13.



Great St. George.

South Underlie 5°.

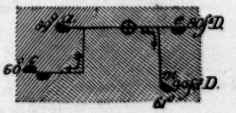
Fig. 14.

Dolcoath.

On Caunter Lode.

South Underlie 12°.

Fig. 15.



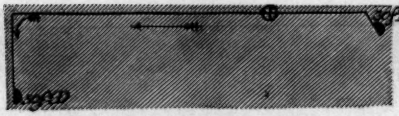
positive. negative.
a with m
e — c
a — c
e — m

Dolcoath.

On Caunter Lode.

South Underlie 12°.

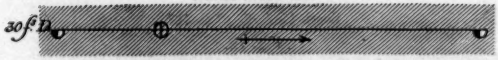
Fig. 16.



On Harriet's Lode.

South Underlie 9°.

Fig. 17.



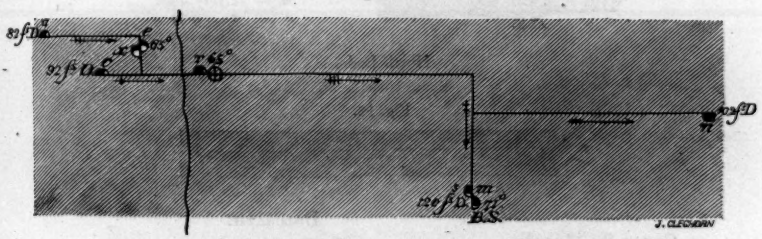
(See intersections of these Lodes, Fig. 27.)

Huel Rose Lead Mine.

Lode 8 feet wide.

South Underlie 25° to 30°.

Fig. 18.



positive. negative.

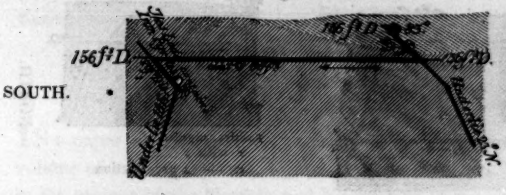
a	with r		strong action
a	—	m or s	ditto
a	—	n	stronger
r	—	m or s	still stronger
r	—	n	ditto
c	—	r	
e	—	r	
x	—	r	
a	—	$(r\,n)$	

Transverse sections of parallel Lodes connected by the Wires, from Fig. 19. to Fig. 26.

POLDICE MINE.

Tin Lode. Old Ore Lode (copper).

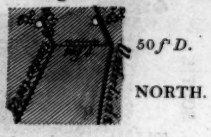
Fig. 19*.



HUEL LEISURE.

Copper Lodes.

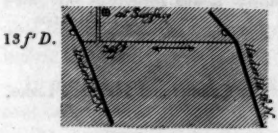
Fig. 20.



HUEL PROSPER.

Copper Lode. Lead Lode.

Fig. 21.



TRESAVEAN MINE.

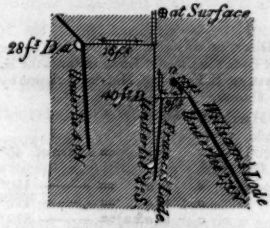
Fig. 22.

positive. negative.
a with *r*
a — *s* more strongly.
r — *s*

HUEL UNITY.

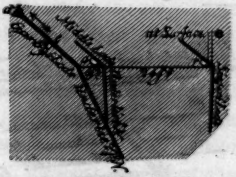
Copper Lodes.

Fig. 23.



positive. negative.
c with *a* slight action
n — *c* stronger
n — *a* ditto

Fig. 24.

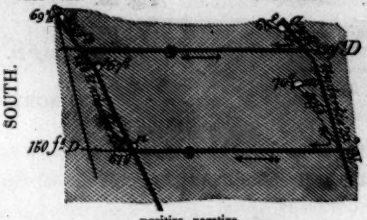


positive. negative.
o with *r* very slight action
e — *r* more
r — *a* great action

* The parts above the bends or angles in the veins or lodes are intended to represent the horizontal directions in perspective. The parts under the angles show their underlie.

Huel Damsel.

Fig. 25.



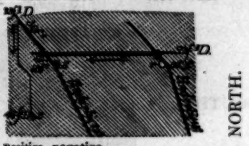
positive. negative.

130 fathoms deep { *a* — *e* slight; *a* — *f* greater; *e* — *f* still more action

150 fathoms deep *m* — *n*

Huel Jewel.

Fig. 26.



positive. negative.

r with *s*

r — *x* great action

Huel Jewel and Huel Damsel Lodes when connected; distance, 130 fathoms; former north of latter.

positive. negative.

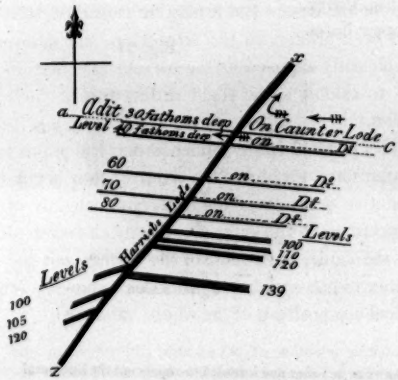
a with *s*

r — *a* very slight action

Dolcoath Mine.

Supposed bird's-eye view of the alternate intersections of the Caunter and Harriet's Lodes, both copper.

Fig. 27.



Direction of electricity (positive) at the adit from the Caunter Lode towards the E. or rather N.E. end of Harriet's, shown by the arrows.

On Harriet's Lode, pos. elect. from *z* to *x* = an arc of 40°

On Caunter Lode . *c* to *a* = —— 190°

c to *x* = -—— 110°

c to (*a*&*x*) = —-— 210°

(*c x*) to *a* = —— 120°

XXVI. *Sequel to a paper on the tendency to Calculous Diseases, and on the Concretions to which such diseases give rise.* By JOHN YELLOLY, *M.D. F.R.S. &c.*

Read June 17, 1830.

IN a paper which the Royal Society did me the honour to publish in the last volume of its Transactions, I gave the analysis of 328 of the calculi contained in the cabinet of the Norfolk and Norwich Hospital; but was prevented from extending my observations over the whole series of specimens, from the circumstance of the remainder not being then divided. Since that period, however, the division of the whole has been effected; and I have therefore been able to complete the analysis, of which I have now the honour to lay the result before the Society.

In my former paper I stated it as probable, that the proportions of the different descriptions of calculi, which formed the undivided part of the cabinet, would not differ materially from that of those which I had analysed; and this proves, in a considerable degree, to be the case. But it may be remarked, that some of the specimens which had been broken in the extraction, and whose interior had been considered as sufficiently exposed for the correct examination of all their laminæ, were found to exhibit some slight differences in their centres, on a more complete division.

I shall adopt the same plan which I pursued in my former paper, and present the results of the analysis in a tabular form, giving, in the order of their occurrence from the centre, the consecutive deposits of the different materials of which the calculi are composed, according to the most prominent character of such material. I shall include in the Table, both those of the former, and the present analysis, amounting together to 663 specimens, in order to present, at one view, a summary of the chemical composition of the whole cabinet *.

* I have placed in the cabinet, a book containing an outline of each calculus, with references; and would also observe, that some of the facts stated in this communication, have been added since it was presented to the Society.

Calculi consisting principally of one deposit.

Lithic acid	164
Lithate of ammonia	55
Oxalate of lime	21
Phosphate of lime	5
Mixed phosphates	35

Calculi consisting of two deposits.

Lithic acid and lithate of ammonia	49
——— and oxalate of lime	10
——— and mixed phosphates	15
——— and phosphate of lime	8
Lithate of ammonia and lithic acid	21
——— and oxalate of lime	63
——— and mixed phosphates	22
——— and phosphate of lime	9
Oxalate of lime and lithic acid	15
——— and lithate of ammonia	3
——— and mixed phosphates	20
——— and phosphate of lime	7
——— and silex	1
Mixed phosphates and oxalate of lime	1
——— and phosphate of lime	2
Phosphates of lime and mixed phosphates	3
——— and oxalate of lime	1

Calculi consisting of three deposits.

Lithic acid, oxalate of lime, and phosphate of lime	2
——— oxalate of lime, and lithate of ammonia	4
——— oxalate of lime, and lithic acid	5
——— lithate of ammonia, and oxalate of lime	2
——— lithate of ammonia, and lithic acid	2
——— lithate of ammonia, and mixed phosphates	2

Lithic acid, oxalate of lime, and mixed phosphates 3
Lithate of ammonia, oxalate of lime, and mixed phosphates . . . 13
——— oxalate of lime, and phosphate of lime 13
——— oxalate of lime, and lithic acid 16
——— oxalate of lime, and lithate of ammonia 7
——— phosphate of lime, and lithate of ammonia 1
——— phosphate of lime, and lithic acid 1
——— phosphate of lime, and oxalate of lime 1
——— phosphate of lime, and mixed phosphates 4
——— lithic acid, and mixed phosphates 6
——— lithic acid, and lithate of ammonia 1
——— lithic acid, and phosphate of lime 4
——— lithic acid, and oxalate of lime 3
Oxalate of lime, lithic acid, and lithate of ammonia 3
——— lithic acid, and oxalate of lime 3
——— lithic acid, and mixed phosphates 5
——— lithic acid, and phosphate of lime 1
——— lithate of ammonia, and phosphate of lime 3
——— lithate of ammonia, and oxalate of lime 2
Mixed phosphates, phosphate of lime, and mixed phosphates . . . 1

Calculi consisting of four deposits.

Lithic acid, lithate of ammonia, lithic acid, and lithate of ammonia . 1
——— oxalate of lime, lithate of ammonia, and phosphate of lime . 1
——— oxalate of lime, lithic acid, and oxalate of lime 1
——— oxalate of lime, lithic acid, and lithate of ammonia . . . 2
Lithate of ammonia, oxalate of lime, lithate of ammonia, and mixed phosphates 5
——— oxalate of lime, lithate of ammonia, and oxalate of lime . . 3
——— oxalate of lime, mixed phosphates, and oxalate of lime . . 2
——— oxalate of lime, lithic acid, and lithate of ammonia 1
——— oxalate of lime, phosphate of lime, and mixed phosphates . 1
——— oxalate of lime, lithic acid, and mixed phosphates 1
——— oxalate of lime, lithic acid, and oxalate of lime 1

Lithate of ammonia, oxalate of lime, lithate of ammonia, and lithic acid	1
——— phosphate of lime, oxalate of lime, and lithate of ammonia	1
Oxalate of lime, lithic acid, lithate of ammonia, and lithic acid	1
——— lithic acid, oxalate of lime, and phosphate of lime	1
——— lithic acid, oxalate of lime, and mixed phosphates	1
——— lithic acid, lithate of ammonia, and mixed phosphates	1
	663

The general results of the annexed Table, differ in no very material respect from those mentioned in my former paper. The great preponderance of calculi of lithic acid, and lithate of ammonia, or of their nuclei; and the usual presence of carbonate of lime, with phosphate of lime, and the mixed phosphates, are prominent characteristics of the whole cabinet. But it will be seen, with interest, that *silex* enters into the composition of one specimen; and as this is a point of great rarity, I shall state to the Society the circumstances under which it exhibited itself, and the mode by which its existence has been ascertained.—In examining a dark brown calculus of oxalate of lime, of about five grs. in weight, which was removed many years since from a boy of nine years of age, I found some minute, colourless, transparent crystals, diffused irregularly in the substance of the dark oxalate, which, from their great hardness, and their insensibility to all the usual reagents, I suspected to be siliceous. The specimen was however so small, as to make it important to establish its nature by one set of experiments only; and I therefore, in my first subsequent visit to London, in February last, requested a valuable member of this Society, Mr. Faraday of the Royal Institution, to examine it with me; which he obligingly did; and the following is an account of the experiments which he employed.

A portion of the calculus being separated, which contained about nine of these granules, the oxalate of lime, and whatever other substance might be in combination with it, was destroyed by heat, and afterwards by muriatic acid. The granules were then left transparent and colourless; capable of scratching glass and agate, and unaffected by nitric or muriatic acids. These granules were then dried and exposed to heat, with a fused mixture of carbonate of

soda and potash. They gradually dissolved, evolving carbonic acid; and a solution of the mass, when cold, being made in water, and neutralized by muriatic acid, gelatinized silica was thrown down from it. A slight excess of muriatic acid was then added, and the whole evaporated to dryness. After withdrawing the muriate of potash and soda by distilled water, the silica was left in its usual white, insoluble state. By comparing the magnitude of these granules with some which were taken from a sand bath, it was calculated that they did not average more than the 400th part of a grain in weight. The granules were thus unequivocally proved to be of silex; and as they were imbedded in, and diffused through, oxalate of lime, a substance of known urinary origin, it is impossible to avoid the conclusion, that the production, or deposition of these two substances, went on simultaneously.

I am the more particular in mentioning the circumstances under which the siliceous deposit exhibited itself, because much discrimination is occasionally necessary, on the part of medical men, to prevent their being deceived, by the mistakes of patients, or their friends, in matters of an unusual nature. And as if the love of exciting surprise and admiration by the marvellous, were not confined to the traveller, there is sometimes found in patients, however singular the fact may appear, an inclination to impose on their professional attendants, by the description or exhibition of something strange and anomalous*.

There are only three instances on record, as far as I know, of the existence of silex in urinary calculi. Two are mentioned by MM. Fourcroy and Vauquelin as occurring among 600 calculi which they analysed; and here the silex was found blended with oxalate of lime, as in the specimen which I have mentioned. The third was observed by Prof. Wurzer, and its principal ingredients were phosphate of lime and lithic acid, the weight of the calculus being 870 grains, and the quantity of silex being one per cent.†. In none of these calculi, however, is the magnitude of the siliceous particles stated.

The deposition of siliceous gravel is mentioned by Dr. Venables of Chelmsford, in a communication published in the Quarterly Journal of Science, Lite-

* Portions of coal, brick, common gravel, and sea shingle, have occasionally been produced, as of urinary origin.

† Annales de Chimie, tome lx. p. 313.

rature, and Art, for Oct.—Dec. 1829*; and the correctness of that gentleman's observations, receives a strong and important confirmation by the instance which I have brought forward; though they might, at first view, be considered as liable to some doubt, both from the circumstance of the granules being uncombined, and from its being necessary to depend, to a certain extent, on the fidelity of the patient as to their source.

Dr. Venables has kindly allowed me the inspection of his specimens, which bear some resemblance (though they are much more minute, and are of an amber tinge,) to those which I have mentioned as coming under my own view; and he has stated in a letter to me, that in one instance, after carefully filtering and putting aside for a fortnight, a portion of the urine from which some of the granules mentioned by him had been derived, he found the inside of the glass studded, in two or three places, with minute crystals of silex, strongly resembling those which were thrown down by the urine. The precise modes in which silex is capable of being held in solution, have not all of them been distinctly ascertained; but this fact bears a considerable analogy to the deposition of regular crystals of rock crystal, from solutions of silex in fluoric acid, or in alkalis, after such solutions have been put aside for a considerable period.

I have not much to add to the statistical observations which I made in my former communication. I may remark, however, that it appears, from information lately obtained by Mr. Copland Hutchison, that the calculations relative to the tendency to calculous diseases in Scotland, in which I followed Mr. Smith of Bristol, have been a good deal under-rated; and that the average disposition of that part of the kingdom to such complaints, differs but little from that of England in general†.

There seems to be much of the same variation with regard to the prevalence of calculous diseases in Scotland, that there is in England; some districts being exceedingly liable to these complaints, while others are very free from

* New Series, No. XII. p. 234.

† Further Inquiry into the comparative Infrequency of Calculous Diseases among Sea-faring People; with some Observations on their Infrequency in Scotland. Medico-Chirurgical Transactions, vol. xvi. p. 94.

them. My own inquiries, in addition to the valuable information communicated by Mr. Hutchison, in the paper already referred to, enable me to state the proportional frequency of the disease in a few districts.

The operations which have been performed at the Dundee Hospital, on cases belonging to Dundee, and to the county of Forfar generally, are about 54 in 36 years, or 1.38 per annum*; this, as the population is 113,000, is at the rate of one case for 107,000 inhabitants.

In the Aberdeen Hospital, the proportion is much more considerable; for in the course of 77 years, as by a list which Mr. Cromar of that establishment was so good as to transmit to me, 285 operations have occurred, on cases belong to Aberdeenshire, and to the town of Aberdeen, containing together, a population of 155,000; which is at the rate of one case for every 42,000 inhabitants. On the other hand, in the Hospital of Inverness, not more than 5 operations, as Mr. Hutchison states, have occurred in the last 20 years, which, for the population of 90,000 contained in Inverness-shire, including the town of Inverness, is not at the rate of more than one for every 300,000 inhabitants. But when it is considered, that this hospital is resorted to by the poor of the contiguous counties of Ross and Nairn, as well as by those of Cromarty and Sutherland, the extreme paucity of stone cases, in this mountainous district, must be manifest.—In the Infirmary of Glasgow, 49 cases have occurred, in the course of 15 years, according to a list which Dr. Macfarlane did me the favour to forward to me, or about 3.26 per annum; and of these, 31 belonged to the city and suburbs, containing about 147,000 inhabitants, which is at the rate of 2 per annum, or one for every 71,000 inhabitants. The remaining 18 were from the country; but I am unable to state the districts from which they were derived. There is reason, however, to suppose, that the tendency of the neighbouring counties to this disease, is very much less than that of Glasgow, for they have only afforded as small a number of calculous diseases as 18 to the Infirmary of that city, which may be regarded as the principal establishment for cases requiring capital operations in the West of Scotland.

In the neighbouring town of Paisley, consisting of 38,000 inhabitants, about 18 cases have occurred in 10 years, all of them of poor inhabitants of

* Observations on the Operation of Lithotomy, by John Crichton, Esq. Edin. Med. & Surg. Journ. vol. xxix. p. 225.

the town, which is at the rate of 1.8 per annum, or one for every 21,000 inhabitants.

In the Infirmary of Edinburgh, 41 cases have occurred in the last 10 years; but I have not been able to procure, separately, the numbers which were derived from the city and the country respectively. Taking, however, the proportions as similar to what is found to be the case in the Glasgow Hospital (which is probably not far from the truth), there would be 24 of that number belonging to the city of Edinburgh, including Leith, and containing a population of 138,000, which would be at the rate of 2.4 per annum, or one case for every 57,000 inhabitants.

In the southern and south-west parts of Scotland, as well as in the northern, the disease is exceedingly rare; for Dr. Craigie of Edinburgh informs me, that it is hardly known in the Dumfries Hospital, which, as being the only establishment of this kind south of Edinburgh and Glasgow, takes in a very large district in that part of the kingdom; and Sir George Ballingall, the Regius Professor of Military Surgery in the University of Edinburgh, states to me, on the most respectable authority, that in Kelso and its neighbouring district, calculous complaints scarcely ever occur.

The county of Northumberland, and that part of the county of Durham which is contiguous to the Tweed, a good deal resemble, in the unfrequency of calculous diseases, the adjoining districts of Scotland; for by a list which was obligingly transmitted to me by Dr. Headlam of Newcastle, it appears, that 95 cases of stone operation occurred in the Newcastle Infirmary during the last 30 years, which is at the rate of 3.6 per annum. Of this number, 64 belonged to the above district, including Newcastle, with the addition of Gateshead, which lies on the opposite bank of the Tyne, in the county of Durham; and these afforded 2.13 cases per annum, which, as the population was 213,000, gave one case for every 100,000 inhabitants. But if the country district be taken without Newcastle or Gateshead, there will then be 29 cases in 30 years for a population of 166,000, and one case for every 172,000 inhabitants.

As very little has been hitherto known concerning the proneness to calculous diseases in Ireland, though it is generally believed that such complaints are unfrequent, I have lately instituted inquiries on the subject, at the various

hospitals in that kingdom, to which Sir CHARLES FLINT, the resident Under-Secretary of State for Ireland in this country, has obligingly given me every facility. From the result of those inquiries it appears, that calculous diseases in Ireland are very rare, particularly in its country population. In various extensive districts from which I have been favoured with returns, stone is entirely unknown; and in others, it occurs with extreme unfrequency. Thus in the counties of Antrim, Armagh, Londonderry, Donegal, Fermanagh, Tyrone, Carlow, Kildare, Kilkenny, and Longford; in King's County; and in the counties of Louth, Wicklow, Clare, Kerry, Galway, Roscommon, Tipperary, and Mayo, containing, together, a population of above three millions and half of persons, not a single operation of lithotomy has occurred in any of their respective hospitals since their establishment; nor has one example, among the poor of those extensive districts, come within the cognizance of the eminent and well-informed practitioners, who have done me the favour of replying to the queries which I transmitted to them on the subject.

In the counties of Down, Monaghan, Leitrim, Sligo, Limerick, and Waterford, and in Queen's County, the population of which amounts together to about 1,200,000 persons, 9 cases of stone operation only have occurred, during the whole time to which the records of the hospitals, or the information or inquiries of their medical officers extend, and which embrace a period hardly short of 40 years. This is at the rate of not more than 0.25 per annum, or one case in 4 years.

In the city and county of Cork, containing, together, above 800,000 inhabitants, about 13 operations of lithotomy have been performed in the last 18 years, or about 0.66 per annum, of which 10 occurred to Dr. WOODROFFE of the South Hospital.

In the hospitals of Dublin, including the Meath or county of Dublin Hospital, it appears from information with which I have been favoured by Mr. RONEY, late President of the Royal College of Surgeons of Dublin, and Mr. CRAMPTON, the Surgeon-general of Ireland, that about 6 cases occur in the course of the year; and this estimate is confirmed by Mr. CARMICHAEL's opinion, as stated by Mr. HUTCHISON.

Making a suitable allowance, therefore, for those counties from which I have

been disappointed in not yet receiving returns*, it does not appear that more than 8 operations of lithotomy occur annually among the poor of Ireland, the whole population of which was considered, in 1821, to be about seven millions†. But when we refer 5 of those cases to the city and county of Dublin, containing, together, a population of about 350,000, (which, from the documents with which I have been favoured, seems to be very near the truth,) it will be seen how minute the proportion of calculous cases must be in the remaining population of Ireland, when not more than 3 operations of lithotomy occur, annually, among the poor of a population of between six and seven millions.

I have mentioned in my former paper, a suspicion, from some facts there stated by me, that the principal occurrence of calculous diseases is in towns. This idea is strengthened by a consideration of the lists of stone cases, for which I have been indebted to Dr. Headlam of Newcastle, and Mr. Eddison of Nottingham. In the town of Newcastle and Gateshead, calculous cases have occurred, within the last 30 years, in the proportion of about one for every 40,000 inhabitants; while in the county of Northumberland, independently, together with that part of the county of Durham which borders on the Tweed, the proportion, (as I have already remarked,) has only been one for 172,000 inhabitants. —In the town of Nottingham, the proportion, in 48 years, has been at the rate of one for every 67,000 inhabitants; while in the county of Nottingham only, it has been one for every 146,000 inhabitants.

In the town of Dundee, the calculous operations have occurred in the proportion of one to 41,000 inhabitants; yet in the county of Forfar, in which Dundee is situated, the proportion has been one to 82,000.

In Glasgow, the proportion has been, in the last 15 years, about one in 71,000; in Paisley, in 10 years, one in 21,000; and in Edinburgh, in a similar time, one in 57,000; while there is every reason for supposing (though I cannot speak from direct evidence upon the subject) that the stone cases afforded to any of those establishments, by country population, is exceedingly small. It is to be remarked, however, that the proportional numbers furnished by the town and country population of Aberdeenshire, during a period of 70 years, are nearly

* These are Cavan, Meath, Westmeath, and Wexford.

† Statistical Illustrations, 3rd edition, 1827.

alike; or rather, that those of the country population have somewhat preponderated; for while Aberdeen has afforded to the amount of one case for 44,000 inhabitants, the proportion of Aberdeenshire alone, without any obvious cause for such difference, has been one for 40,000 inhabitants.

In Ireland, the great preponderance of calculous cases originating in towns, is likewise strongly marked; for in Dublin, to judge from the returns of the county hospital for the last 12 years, two thirds of the cases, or 4 per annum, seem to be furnished by that city, which is about one half of the number afforded by the whole remaining part of Ireland, and is at the rate of one for every 45,000 inhabitants. In Cork likewise, the proportions are much less than those of Dublin, being about one for 160,000 inhabitants; yet this infinitely exceeds the usual product, either of the county of Cork, or of Ireland generally.

Where the circumstances which have a tendency to produce calculous diseases, are so very obscure, so difficultly traceable, and so full of anomalies, I have thought it useful to notice the local situations which are remarkable either for the frequency or unfrequency of such maladies, because the attention of observers may thus be directed to analogies, or discrepancies, which may not before have been sufficiently the subject of remark. That there are certain affections of the digestive organs which favour the occurrence of calculous complaints, is very generally admitted; and that these affections are likewise exceedingly prevalent in towns, and more especially among persons who practise sedentary occupations, is likewise well known. But while dyspeptic complaints are so common, that they form a very large share of the diseases which present themselves in the medical charities, both of London and the country, it still remains a problem, to what particular circumstances of constitution or habit, the origin of stone is to be attributed, which, in comparison of other chronic maladies, is so rare; and what are the peculiarities which render some dyspeptics liable to the complaint, and thousands of others exempt from it.

That there is a certain connection between calculous tendency, and some prevailing diathesis of towns, is supported by the greater frequency of stone in town, than country populations; and I was inclined to consider it not improbable, that this diathesis might be the scrophulous, both from mesenteric diseases

prevailing so much among the children of crowded cities, and from scrophulous affections being very common in Norfolk. Still, however, the same difficulty presents itself, as occurs with regard to affections of the digestive organs; namely, why so minute a portion of scrophulous persons should be affected with this disease, and so large a number escape. I am disposed, indeed, to consider the idea of a probable connection between a scrophulous and calculous diathesis, as very much weakened by the consideration, that in many of those parts of Ireland, where calculous complaints are hardly at all known, scrophulous affections are stated to me to be exceedingly common.

I must still confess the difficulty of referring the disposition to calculous affections, to any known circumstances of air, water, soil, or habits of life. The eastern position of Norfolk and Suffolk, produces more than the average bleakness of other parts of the kingdom, particularly in the spring months; but this inclemency is exceeded, both in the northern parts of the island, and the more elevated parts of Ireland. The poorer part of the population have much of a farinaceous aliment, often not well fermented: but still the main article of food is fine wheaten bread, which has been, for a long period, in such common use over the whole of England. In Scotland, too, the latter is, as I am informed, fast superseding the bread made of the coarser grains.

In Ireland, the diet of the humbler classes of society is very much confined to potatoes, and butter-milk, or skim-milk; often, unfortunately however, without either of the latter; and it appears highly probable that this diet, and perhaps, to a certain extent, the use of ardent spirits, is unfavourable to the formation or deposition of lithic acid, with which the tendency to calculous diseases is much connected. The freedom from this formidable class of complaints seems, however, to be dearly purchased, by the want of many of the comforts which the labourers of England and Scotland possess; nor is it accompanied by a less than usual tendency to many of the most ordinary complaints of Great Britain, as pulmonary consumption and scrophula, which are mentioned by several of my intelligent correspondents, as being very frequent among the Irish commonalty. The effect of a town residence, in producing a tendency to calculous complaints, is very strongly exemplified in Dublin, where they are so much more prevalent than in other parts of Ireland. This circumstance will, I venture to hope, meet with the particular attention of the Dublin

faculty, who unite, in so a high a degree, the requisites for prosecuting a difficult professional investigation.

The middle and active period of life, laborious and healthful occupations, together with a fair opportunity of obtaining nutritive aliment, and ordinary comforts, seem best adapted for producing a freedom from chronic ailments, and among them, from urinary calculus.

Mr. HUTCHISON has shown, how unfrequent the latter is among sailors; and there is every reason for supposing that this freedom is, in a considerable degree, participated by the military profession. On this point I have thought it desirable to institute some inquiries, the result of which I shall have the honour to lay before the Society.

In a valuable report published by Sir JAMES MACGRIGOR, on the diseases of the British army in the peninsula, under the command of the Duke of WELLINGTON, no case of calculus appears to have presented itself during the period of which he treats, viz. between December 1811 and June 1814, though above 330,000 cases were admitted into the general and regimental hospitals during that period*.

In the last 15 years, Sir JAMES informs me, that 4 cases only of calculus have occurred in the English army in Britain; and Mr. CRAMPTON, the surgeon-general of Ireland, states, that one example of lithotomy only, in which the operation was performed by himself, has occurred, within the same period, in the army in Ireland. I am, however, able to add to this, on the authority of Dr. PITCAIRNE of Cork, the case of an officer of the Scots Greys, whom I had occasion to visit at the barracks here, who was operated upon at Cork, by Dr. WOODROFFE, about two years since, in his way to join his regiment. Mr. CRAMPTON likewise informed me, on the authority of Sir JAMES WYLIE, physician to the late, and to the present Emperor of Russia, that calculous diseases are hardly known in the Russian army.

The Baron DELESSERT of Paris, has done me the favour to procure from the Baron LARREY, and M. GAMA, surgeons-in-chief to the great military hospitals of Gros Caillou, and Val de Grace, in the French metropolis, a report as to the prevalence of calculous complaints among the French soldiery. The Baron

* Sketch of the Medical History of the British Armies of the Peninsula of Spain and Portugal during the late Campaigns. Medico-Chirurgical Transactions, vol. vi. p. 381.

Larrey states, that in the course of 30 years, only 5 operations of lithotomy have been performed at the Gros Caillou, (4 of which were on soldiers, and one on a soldier's child,) and one operation at the Val de Grace. M. Gama states, that during 6 years that he has been surgeon-in-chief of the military hospital of Val de Grace, and 8 previously, that he exercised the same functions in the military hospital at Strasbourgh, he has not once had occasion to perform the operation of lithotomy. He mentions likewise, that the disease is very rare in the army generally; and that no case of stone operation has occurred to him, during any part of his extensive military service.—It is very probable, however, that as the circumstances which concern health can be more particularly guarded in the naval service, than the military, according to the very judicious observations of Sir James Macgrigor upon this subject, there may be, upon the whole, a less liability to disease in the former, than in the latter.

Before closing my observations, I cannot forbear expressing my regret, that since the introduction of lithotrity by M. Civiale, as a succedaneum for the operation of lithotomy, the beneficial effects of that practice do not seem to have been completely established in this country; though it has been recommended by the singular dexterity, and the conciliating deportment, of the Baron Heurteloup and Mr. Costello. It is highly to the credit of our principal metropolitan surgeons, that all of them with whom I have conversed on the subject, are anxious that this plan of removing a calculus should have a fair trial; and I trust that in a matter in which the interests of humanity are so intimately concerned, such attention may be speedily given to the subject, by those who are qualified to direct public opinion, as may lead to a proper appreciation of the merits of the practice, the circumstances under which it may be best exercised, and the mode by which the manual dexterity which it requires, may be most readily obtained.

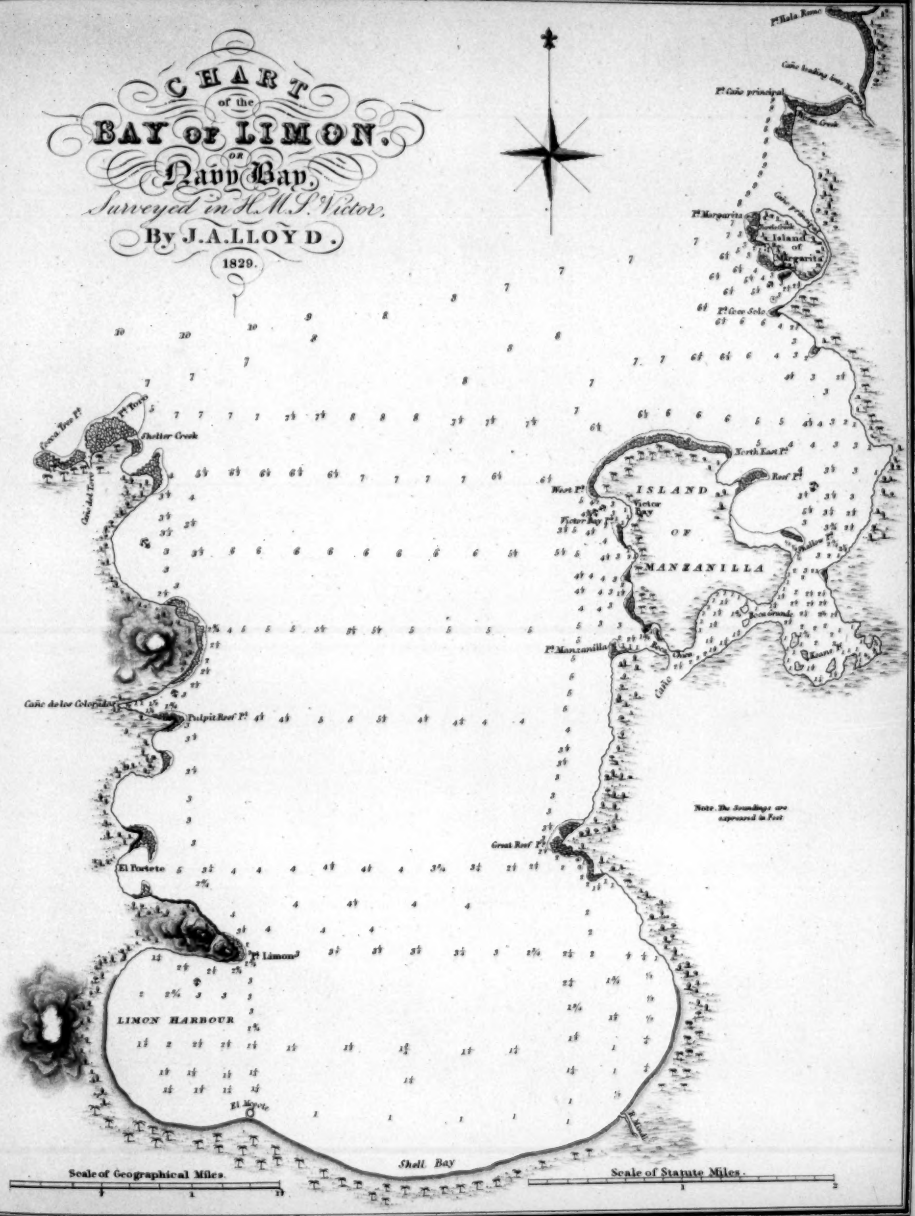
Carrow Abbey, near Norwich,
June 16, 1830.

CHART of the BAY OF LIMON, or Navy Bay,

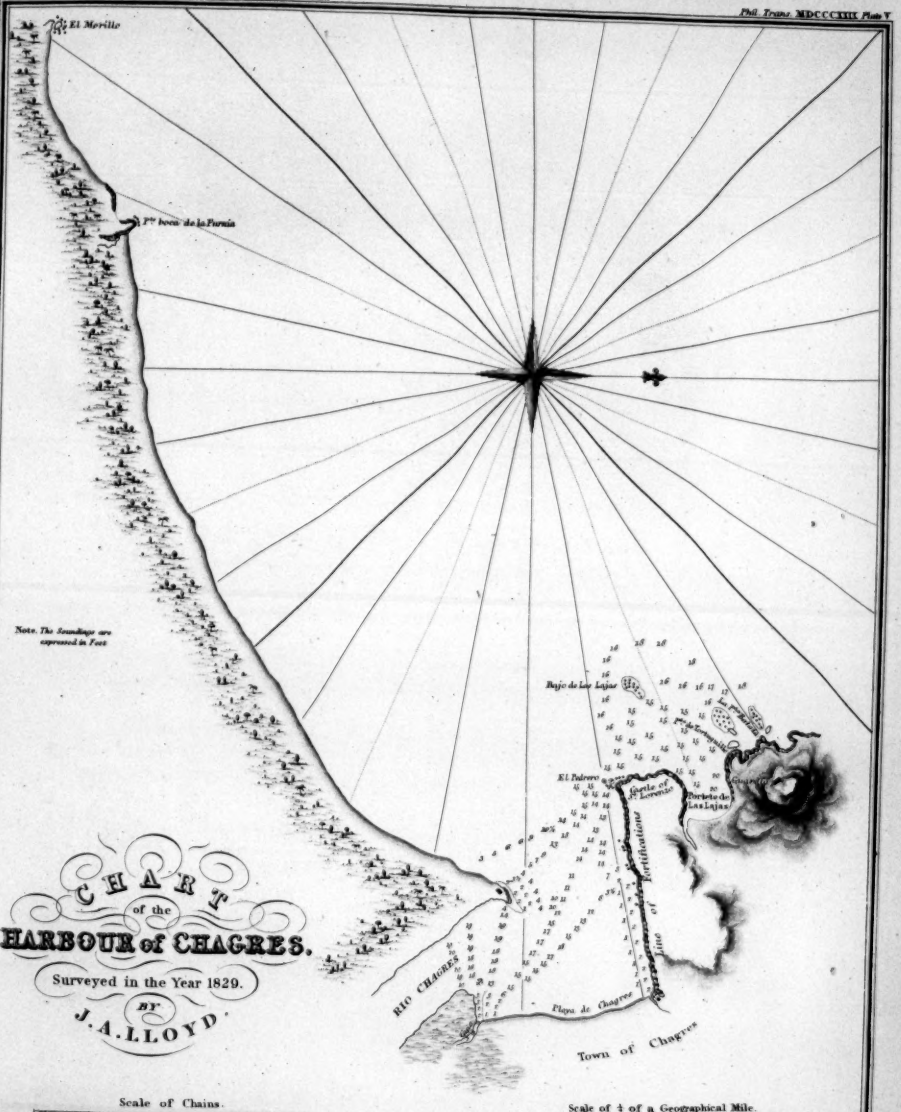
Surveyed in H.M.S. Victor,

By J.A. LLOYD.

1829.



Engraved by J. Gardner.



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MDCCCXXX.

METEOROLOGICAL JOURNAL,

KEPT BY THE ASSISTANT SECRETARY

AT THE APARTMENTS OF THE

ROYAL SOCIETY,

BY ORDER OF

THE PRESIDENT AND COUNCIL.

MDCCCXXX.

METEOROLOGICAL JOURNAL FOR JANUARY, 1830.

1830. January.	9 o'clock, A.M.		3 o'clock P.M.		Dew Point at 9 A.M. In degrees of Fahr.	External Thermometer.				Rain, in inches. Read off at 9 A.M.	Direction of the Wind at 9 A.M.	Remarks.
	Barom.	Attach. Therm.	Barom.	Attach. Therm.		Fahrenheit.		Self-registering.				
						9 A.M.	3 P.M.	Lowest.	Highest.			
☿ 1	30.631	31.5	30.605	33.2	27	31.8	33.5	28.5	33.5		SE	Overcast—light wind.
♄ 2	30.510	32.8	30.486	34.0	24	30.8	32.8	29.4	32.8		NE	Overcast—light brisk wind.
☉ 3	30.496	34.9	30.412	36.2	29	34.9	36.0	29.5	36.0		S	Overcast and foggy.
☽ 4	30.273	36.0	30.237	37.0	29	37.2	37.8	31.8	37.8	0.236	NE	Foggy—light rain.
♂ 5	30.257	36.9	30.237	38.5	35	36.0	40.5	34.3	40.5	0.194	SW	Overcast and foggy—light wind.
☿ 6	30.367	35.3	30.248	36.4	31	31.4	35.0	28.7	38.7		NW	Overcast and foggy—light wind.
♃ 7	29.805	38.6	29.834	40.5	38	39.7	40.8	30.3	42.7	0.017	NW	A.M. Foggy. P.M. Fine—light haze.
☿ 8	30.055	37.3	30.039	38.5	33	35.3	36.8	33.3	36.8	0.008	NNE	Snow, early A.M. Overcast—light wind.
○ ♄ 9	30.179	36.8	30.033	38.6	32	34.3	36.7	32.2	38.7	0.014	NNW	Lightly cloudy and hazy.
☉ 10	29.781	37.6	29.759	38.2	22	35.4	37.3	33.3	37.3	0.019	NW	Clear and cloudless—light brisk wind.
☽ 11	29.690	36.3	29.576	37.8	27	35.1	36.0	33.3	36.4		WNW	Fine—light clouds and brisk wind. Snow at noon.
♂ 12	29.784	34.7	29.875	36.2	29	31.9	33.0	30.4	33.0	0.025	N	Overcast—snow in large flakes.
☿ 13	29.880	33.7	29.242	34.4	27	31.6	31.8	29.6	31.8	0.081	NNE	Snow—light wind.
♃ 14	29.878	31.4	29.885	31.4	24	29.2	29.0	25.4	31.7	0.011	ENE	Overcast—light wind.
☿ 15	29.909	33.7	29.842	34.8	30	32.6	34.8	27.7	34.8		E	Overcast—light rain and wind.
♄ 16	29.907	33.4	29.944	34.2	30	31.7	30.8	28.3	32.4	0.033	ENE	Overcast—light rain with brisk wind.
☉ 17	29.880	30.2	29.883	30.7	16	24.7	24.7	21.3	24.7	0.019	E	Overcast.
☽ 18	29.970	27.8	29.911	27.2	19	21.3	23.8	20.3	23.8		NE	Lightly cloudy and foggy.
♂ 19	29.688	25.3	29.557	27.2	19	20.4	27.5	17.2	31.4		ESE	Hazy—light wind. Strong fog A.M.
☿ 20	28.727	30.0	28.631	33.2	30	32.7	35.5	19.4	35.5		ESE	Hazy—light wind. Snow early A.M.
♃ 21	29.353	32.3	29.435	34.8	32	32.5	35.8	28.7	36.0	0.406	SW	Fine—light haze and wind.
☿ 22	29.590	37.5	29.641	37.9	34	35.3	36.2	31.6	36.2	0.017	ESE	Overcast—light wind. Rain A.M.
♄ 23	29.872	37.3	29.847	37.0	32	32.7	35.4	31.3	35.4	0.097	N	A.M. Snow. P.M. Rain—light wind.
● ☉ 24	29.888	36.8	29.933	38.9	34	35.4	38.7	31.7	38.7	0.036	E	Foggy.
☽ 25	30.250	37.8	30.307	39.5	33	35.5	38.4	34.2	38.4		N	A.M. Hazy. P.M. Fine and clear.
♂ 26	30.316	38.3	30.232	38.5	34	35.7	37.0	33.6	37.0		E	Overcast and foggy—light wind.
☿ 27	29.873	38.6	29.787	39.2	35	35.3	37.5	34.2	37.5		NW	A.M. Foggy—snow. P.M. Light rain.
♃ 28	30.076	39.6	30.134	39.5	35	35.3	36.7	33.6	36.7	0.278	NNE	Overcast—light haze and wind.
☿ 29	30.211	39.5	30.202	39.8	35	35.3	35.6	33.6	35.6	0.167	NNE	Overcast and foggy—light wind.
♄ 30	29.949	38.3	29.888	39.6	34	35.0	38.2	31.6	38.2		NNW	Overcast. A.M. Snow, early. P.M. Light rain.
☉ 31	30.083	31.3	30.059	30.8	17	22.6	23.6	21.4	23.6	0.131	ENE	A.M. Snow. P.M. Fine—light clouds and wind.
	Mean	Mean	Mean	Mean	Mean	Mean	Mean	Mean	Mean	Sum		
	29.970	34.9	29.926	35.9	29.2	32.5	34.4	29.3	35.0	1.919		

Monthly Mean of the Barometer, corrected for Capillarity and reduced to 32° Fahr. ..... { 9 A.M. 29.966 | 3 P.M. 29.919 }

OBSERVANDA.

Height of the Cistern of the Barometer above a Fixed Mark on Waterloo Bridge ..... = 83 feet 2½ in.

..... above the mean level of the Sea (presumed about) ..... = 95 feet.

The External Thermometer is 2 feet higher than the Barometer Cistern.

Height of the Receiver of the Rain Gauge above the Court of Somerset House ..... = 79 feet 0 in.

The hours of observation are of Mean Time, the day beginning at Midnight.

The Thermometers are graduated by Fahrenheit's Scale.

The Barometer is divided into inches and decimals.

METEOROLOGICAL JOURNAL FOR FEBRUARY, 1830.

1830. February.	9 o'clock, A.M. Barom.	9 o'clock, A.M. Attach. Therm.	3 o'clock, P.M. Barom.	3 o'clock, P.M. Attach. Therm.	Dew Point at 9 A.M. in degrees of Fahr.	External Thermometer. Fahrenheit. 9 A.M.	3 P.M.	Self-registering. Lowest.	Highest.	Rain, in inches. Read off at 9 A.M.	Direction of the Wind at 9 A.M.	Remarks.
☽ 1	29.898	26.7	29.832	27.8	17	22.6	25.2	20.1	25.2	0.014	NNE	Fine—cloudy—light wind.
♂ 2	29.809	25.6	29.789	26.2	17	20.3	20.0	18.6	20.6		E	Continued snow—light wind.
☿ 3	29.830	24.5	29.843	24.8	21	22.3	21.8	17.7	25.6		N	Light snow, with brisk wind.
♃ 4	29.848	25.3	29.775	29.2	20	26.8	30.0	18.7	30.4		NNE	Cloudy and hazy—light wind.
♀ 5	29.735	25.8	29.666	28.2	18	22.3	24.5	18.7	24.5		NNE	Snow—light wind.
♄ 6	29.575	23.3	29.581	24.6	17	19.6	20.8	15.3	28.8		N	Overcast and foggy—light wind.
☌ ☉ 7	29.400	26.7	29.369	30.3	30	30.7	37.4	18.5	45.6		SSE	Overcast—light rain.
☽ 8	29.412	36.2	29.613	42.8	44	44.8	46.4	29.4	46.4		SW	A.M. Lowering. P.M. Cloudless—light brisk wind.
♂ 9	29.345	41.3	29.519	43.4	45	45.2	44.8	40.2	45.8	0.389	SSW	High wind with rain, early A.M.—overcast—light rain.
☿ 10	29.807	39.6	29.938	42.2	36	36.6	41.6	33.3	41.6		W	A.M. Foggy. P.M. Fine and clear.
♃ 11	30.173	38.7	30.175	42.0	36	36.7	43.2	31.2	43.2		W	Lightly overcast.
♀ 12	30.146	42.6	30.129	45.0	37	37.7	41.5	35.5	42.3		S	Fine—light clouds and wind.
♄ 13	30.170	40.4	30.178	42.5	34	34.7	40.2	32.5	40.2		SSE	Fine—light wind and haze.
☉ 14	30.100	38.5	30.095	39.3	32	32.3	36.5	30.2	36.5		SW	Overcast—light fog.
☽ 15	30.314	38.6	30.330	39.4	33	33.7	38.2	31.4	38.2		NE	Overcast—light brisk wind.
♂ 16	30.331	38.6	30.276	38.6	33	33.4	34.8	31.7	34.8		N	Strong fog—light wind.
☿ 17	29.983	36.3	29.829	36.8	30	30.3	33.5	29.3	35.5		NW	Overcast—brisk wind.
♃ 18	29.698	37.3	29.732	39.2	35	35.7	37.0	29.4	37.3		NNE	A.M. Overcast. P.M. Fine—light wind.
♀ 19	29.773	37.7	29.766	40.2	34	34.7	37.0	29.0	37.7		W	Fine—lightly cloudy.
♄ 20	29.837	35.8	29.810	39.2	29	30.3	38.0	26.3	38.0		WSW	Fine—lightly hazy.
☉ 21	29.820	37.3	29.475	41.5	35	37.0	41.0	28.4	41.0		SW	Cloudy—light wind. Rain P.M.
☽ 22	29.698	37.1	29.723	39.8	31	34.6	39.5	30.0	47.7	0.342	WSW	Lightly cloudy. Rain P.M.
● ♂ 23	29.682	42.4	29.785	45.2	48	48.3	50.5	33.7	50.5	0.075	WSW	Rain with light brisk wind.
☿ 24	29.970	46.9	30.062	51.5	49	49.8	55.4	46.7	55.5	0.072	W	Overcast.
♃ 25	30.195	50.3	30.192	53.0	49	49.4	54.2	47.0	54.8		SW	Fine and cloudless.
♀ 26	30.076	53.3	30.068	55.5	50	50.7	53.0	47.7	54.3	0.086	SSW	Light clouds and wind—rain at night.
♄ 27	30.068	53.7	30.071	56.8	51	51.6	55.8	48.4	56.3	0.028	W	Fine—hazy.
☉ 28	30.070	52.7	30.139	55.8	48	48.7	54.6	45.8	55.3		WSW	Fine—lightly overcast.
	Mean	Mean	Mean	Mean	Mean	Mean	Mean	Mean	Mean	Sum		
	29.877	37.6	29.884	40.0	34.2	35.7	39.2	30.9	40.5	1.006		

Monthly Mean of the Barometer, corrected for Capillarity and reduced to 32° Fahr. ..... { 9 A.M. 29.866 } { 3 P.M. 29.867 }

OBSERVANDA.

Height of the Cistern of the Barometer above a Fixed Mark on Waterloo Bridge..... = 83 feet 2½ in.

.....above the mean level of the Sea (presumed about) ..... = 95 feet.

The External Thermometer is 2 feet higher than the Barometer Cistern.

Height of the Receiver of the Rain Gauge above the Court of Somerset House..... = 79 feet 0 in.

The hours of observation are of Mean Time, the day beginning at Midnight.

The Thermometers are graduated by Fahrenheit's Scale.

The Barometer is divided into inches and decimals.

METEOROLOGICAL JOURNAL FOR MARCH, 1830.

1830. March.	9 o'clock, A.M. Barom.	9 o'clock, A.M. Attach. Therm.	3 o'clock, P.M. Barom.	3 o'clock, P.M. Attach. Therm.	Dew Point at 9 A.M. in degrees of Fahr.	External Thermometer. Fahrenheit. 9 A.M.	Fahrenheit. 3 P.M.	Self-registering. Lowest.	Self-registering. Highest.	Rain, in inches. Read off at 9 A.M.	Direction of the Wind at 9 A.M.	Remarks.
☽ 1	30.347	49.8	30.360	53.5	47	48.3	51.4	46.3	51.4		W	Fine—hazy.
♂ 2	30.430	52.3	30.421	54.4	49	49.8	51.5	47.3	51.7		SSW	Lightly cloudy.
☿ 3	30.423	51.6	30.383	54.5	45	48.7	52.2	46.3	52.4		S	Overcast.
♃ 4	30.228	46.7	30.032	49.0	34	39.4	43.0	34.7	43.3		E	Clear and cloudless—brisk wind.
♀ 5	29.981	45.8	29.954	49.5	36	40.3	48.6	34.4	48.6		ENE	Fine and clear—light wind.
♄ 6	30.030	46.3	29.987	50.0	38	42.4	49.5	37.7	50.3		NE	A.M. Light fog. P.M. Fine and clear.
☉ 7	30.005	43.2	30.005	46.3	36	36.7	42.8	32.7	42.8		NNE	A.M. Light fog. P.M. Clear and cloudless—light wind.
☽ 8	30.038	42.6	29.971	46.4	36	36.3	46.4	34.6	46.4		E	A.M. Overcast. P.M. Fine and clear.
○ ♂ 9	29.675	45.3	29.562	46.6	41	43.7	46.8	35.7	46.8		SE	Overcast—light continued rain.
☿ 10	29.643	46.9	29.674	50.4	46	46.3	50.2	42.8	50.6	0.081	NNW	A.M. Rain. P.M. Fine.
♃ 11	29.667	49.7	29.809	53.4	49	51.1	56.8	45.3	57.3	0.039	NW	Light clouds and wind.
♀ 12	30.116	52.3	30.014	51.2	47	50.2	54.8	44.2	55.3		WSW	Fine—lightly cloudy.
♄ 13	30.209	50.3	30.303	53.5	34	44.7	51.6	40.2	51.7		W	Fine—light clouds.
☉ 14	30.167	48.7	29.983	53.5	39	44.3	50.7	37.6	51.3		SW	Fine—lightly cloudy.
☽ 15	29.479	48.6	29.390	50.4	47	47.6	48.5	40.8	48.7		SSW	A.M. Fine—broken clouds. P.M. Rain. Evening clear.
♂ 16	29.634	46.7	29.716	49.5	36	43.3	45.8	35.8	49.8	0.019	WSW	Fine and clear. Rain and hail at intervals.
☿ 17	29.792	47.8	29.930	51.6	48	49.7	54.5	40.7	55.8	0.022	WSW	Overcast—light wind.
♃ 18	30.152	50.7	30.209	53.9	45	52.3	54.6	45.6	55.7		WNW	Light clouds and wind.
♀ 19	30.256	53.3	30.195	55.8	47	50.5	54.6	48.3	54.9		SSW	Overcast.
♄ 20	30.134	52.6	30.174	54.8	37	46.6	52.6	44.4	53.3	0.033	W var.	Fine and cloudless—light wind.
☉ 21	30.373	51.6	30.351	55.4	44	45.5	54.7	41.7	56.3		W	Fine and clear—light clouds.
☽ 22	30.179	51.3	30.021	53.2	43	48.3	49.4	43.6	50.5		SSW	Lowering—rain with light wind.
♂ 23	30.064	50.8	30.040	53.3	36	45.7	51.2	39.4	51.7		WSW	A.M. Overcast. P.M. Fine.
● ☿ 24	30.125	52.8	30.166	54.9	49	52.2	55.8	45.4	56.4		W	A.M. Light fog. P.M. Fine.
♃ 25	30.377	51.5	30.366	56.5	45	48.3	60.8	41.3	61.6		W	Fine—light clouds and wind.
♀ 26	30.561	54.7	30.566	59.6	45	50.7	65.0	43.5	65.7		WSW	Fine and cloudless—faint haze.
♄ 27	30.600	55.7	30.547	60.1	45	51.3	64.8	45.7	65.3		W	Fine and cloudless—light haze.
☉ 28	30.458	53.4	30.385	59.3	42	51.3	62.5	45.4	63.3		N	Clear and cloudless.
☽ 29	30.309	50.2	30.120	57.5	42	43.7	61.8	40.3	63.3		E	Fine and clear—light wind.
♂ 30	30.032	53.7	30.002	58.8	43	51.4	58.3	43.3	61.5		N	Clear and cloudless.
☿ 31	29.792	49.6	29.704	54.0	41	44.6	53.8	40.8	56.1		E	Strong fog—Rain, with brisk wind, at night.
	Mean	Mean	Mean	Mean	Mean	Mean	Mean	Mean	Mean	Sum		
	30.102	49.9	30.075	53.3	42.3	46.6	53.1	41.5	53.9	0.194		

Monthly Mean of the Barometer, corrected for Capillarity and reduced to 32° Fahr. ..... 9 A.M. 30.060 — 3 P.M. 30.025

OBSERVANDA.

Height of the Cistern of the Barometer above a Fixed Mark on Waterloo Bridge ..... = 83 feet 2½ in.

..... above the mean level of the Sea (presumed about)..... = 95 feet.

The External Thermometer is 2 feet higher than the Barometer Cistern.

Height of the Receiver of the Rain Gauge above the Court of Somerset House ..... = 79 feet 0 in.

The hours of observation are of Mean Time, the day beginning at Midnight.

The Thermometers are graduated by Fahrenheit's Scale.

The Barometer is divided into inches and decimals.

METEOROLOGICAL JOURNAL FOR APRIL, 1830.

1830. April.	9 o'clock, A.M. Barom.	9 o'clock, A.M. Attach. Therm.	3 o'clock, P.M. Barom.	3 o'clock, P.M. Attach. Therm.	Dew Point at 9 A.M. in degrees of Fahr.	External Thermometer. Fahrenheit. 9 A.M.	Fahrenheit. 3 P.M.	Self-registering. Lowest.	Highest.	Rain, in inches. Read off at 9 A.M.	Direction of the Wind at 9 A.M.	Remarks.
♃ 1	29.694	46.4	29.708	48.8	37	37.5	42.0	34.9	42.0	0.389	N	Overcast—rain and snow with brisk wind A.M.
♀ 2	29.542	45.7	29.372	45.6	38	38.7	39.8	36.2	39.8	0.250	E	Rain—brisk wind.
♄ 3	29.218	44.3	29.519	45.8	39	39.3	40.0	36.3	40.0	0.583	N	A.M. Rain. P.M. Overcast—brisk wind.
☉ 4	30.112	42.3	30.152	45.7	29	35.3	41.8	30.7	42.3	0.014	NNW	Fine—lightly cloudy.
☽ 5	30.188	40.7	30.085	45.0	29	34.6	46.2	29.7	46.2		N	A.M. Overcast. P.M. Fine.
♂ 6	29.778	43.4	29.751	47.5	31	41.3	56.0	31.4	57.3		W	Fine and cloudless—light haze.
☿ 7	29.878	48.3	29.828	51.8	42	48.4	54.2	40.3	54.7		SSW	Overcast—light wind.
○ ♃ 8	29.620	51.3	29.552	55.0	45	53.3	62.2	45.3	64.3		ESE	Fine—light clouds.
♀ 9	29.433	57.2	29.428	60.6	50	57.4	62.3	48.4	62.7		SSW	Fine and clear—light clouds and wind.
♄ 10	29.479	51.7	29.555	57.8	48	49.6	54.6	45.3	58.7	0.283	W	A.M. Cloudy. P.M. Fine—showers.
☉ 11	29.619	55.9	29.613	58.3	44	52.7	56.6	45.6	56.6	0.008	SSW	Fine and clear—cloudy—light showers.
☽ 12	29.451	54.7	29.492	57.2	52	53.8	54.8	46.8	57.4	0.167	NNW	Cloudy—showers.
♂ 13	29.773	56.3	29.856	57.5	46	53.7	57.0	43.3	57.7		W	Fine and clear—light brisk wind.
☿ 14	30.040	53.5	29.977	58.7	44	52.4	60.2	43.2	61.6		W	Fine—lightly cloudy.
♃ 15	29.792	55.3	29.708	58.2	50	53.4	58.5	49.4	59.2		S	Overcast.
♀ 16	29.674	58.4	29.698	61.0	56	58.5	61.4	51.3	63.3	0.042	WSW	Lightly cloudy—light rain P.M.
♄ 17	29.686	59.2	29.653	61.2	52	56.6	58.6	52.3	61.6	0.008	SSW	Lowering—light brisk wind.
☉ 18	29.860	60.3	29.837	60.6	45	54.0	60.4	44.6	61.6	0.075	W	Fine and clear—light clouds.
☽ 19	29.635	53.7	29.637	59.3	43	52.9	55.6	47.3	59.7		NNW	Overcast—brisk wind. At 4 P.M. thunder and lightning with heavy rain and hail.
♂ 20	29.732	53.8	29.766	57.4	50	50.7	55.2	43.2	58.4	0.133	W	Overcast—rain A.M.
☿ 21	29.898	53.6	29.779	57.8	51	51.2	55.2	43.7	58.6	0.022	WSW	Overcast—light rain.
● ♃ 22	29.567	55.7	.....	.....	52	54.0	.....	50.7	56.7		WSW	Overcast.
♀ 23	29.305	58.5	29.257	60.5	55	55.7	55.5	52.3	57.3		SSW	Light rain—strong high wind at night.
♄ 24	29.342	56.9	29.470	59.2	52	52.3	55.3	49.3	57.7	0.022	WSW	A.M. Rain. P.M. Fine and clear. Brisk wind.
☉ 25	30.017	55.3	30.098	59.3	40	51.7	58.7	40.8	62.5	0.025	W	Fine and clear—cloudy.
☽ 26	30.221	58.6	30.210	60.6	50	56.6	61.8	47.2	64.4		SSW	Fine and clear—light clouds.
♂ 27	30.247	58.6	30.196	61.1	42	57.9	62.8	44.6	63.6		SSE	Very clear and cloudless—light wind.
☿ 28	30.139	59.7	30.078	61.6	50	59.4	64.4	46.6	67.3		ESE	Very clear and cloudless—brisk wind.
♃ 29	30.028	63.7	29.921	66.2	50	62.7	69.5	51.3	70.7		ESE	Very clear and cloudless—light wind.
♀ 30	29.885	65.3	29.849	68.9	54	64.6	70.0	52.3	72.0		E	Very clear and cloudless—light wind.
	Mean	Mean	Mean	Mean	Mean	Mean	Mean	Mean	Mean	Sum		
	29.758	53.8	29.760	56.8	45.5	51.3	56.2	44.1	57.9	2.021		

Monthly Mean of the Barometer, corrected for Capillarity and reduced to 32° Fahr. ..... { 9 A.M. 29.707 } { 3 P.M. 29.701 }

OBSERVANDA.

Height of the Cistern of the Barometer above a Fixed Mark on Waterloo Bridge..... = 83 feet 2½ in.

..... above the mean level of the Sea (presumed about) ..... = 95 feet.

The External Thermometer is 2 feet higher than the Barometer Cistern.

Height of the Receiver of the Rain Gauge above the Court of Somerset House ..... = 79 feet 0 in.

The hours of observation are of Mean Time, the day beginning at Midnight.

The Thermometers are graduated by Fahrenheit's Scale.

The Barometer is divided into inches and decimals.

METEOROLOGICAL JOURNAL FOR MAY, 1830.

1830. May.	9 o'clock, A.M. Barom.	9 o'clock, A.M. Attach. Therm.	3 o'clock, P.M. Barom.	3 o'clock, P.M. Attach. Therm.	Dew Point at 9 A.M. in degrees of Fahr.	External Thermometer. Fahrenheit. 9 A.M.	Fahrenheit. 3 P.M.	Self-registering. Lowest.	Self-registering. Highest.	Rain, in inches. Read off at 9 A.M.	Direction of the Wind at 9 A.M.	Remarks.
♄ 1	30.002	67.3	30.005	66.3	48	59.6	62.6	52.4	66.7		SW	Fine and clear—light clouds and brisk wind.
☉ 2	30.152	65.7	30.128	64.5	46	59.0	64.5	46.7	66.4		W	Fine and clear—light clouds.
☽ 3	30.217	58.8	30.180	61.7	40	55.3	60.4	45.7	61.7		N	Fine—cloudy.
♂ 4	30.171	60.3	30.125	62.2	47	57.6	62.8	43.7	63.7		E	Very clear and cloudless—light brisk wind.
☿ 5	30.087	60.7	30.107	63.6	49	60.4	68.8	46.4	71.6		E	Fine and clear—light brisk wind.
♃ 6	29.933	65.5	29.824	68.2	58	66.5	73.4	55.7	77.3		NNE	Fine—cloudy.
○ ♀ 7	29.631	69.6	29.565	71.6	67	69.3	72.0	58.3	75.7		ESE	Fine and clear—cloudy.
♄ 8	29.597	68.3	29.504	66.3	55	65.3	60.2	54.7	67.3		ESE	A.M. Fine. P.M. Heavy showers.
☉ 9	29.332	63.1	29.312	62.3	51	54.7	53.6	52.7	53.8	0.264	SSW	Lowering—light rain.
☽ 10	29.411	58.7	29.511	60.4	47	51.7	51.8	47.8	55.7	0.217	WSW	Overcast—light rain.
♂ 11	29.699	53.7	29.732	56.3	42	47.6	50.8	42.6	51.6	0.061	N	Overcast.
☿ 12	29.781	52.8	29.805	57.4	44	48.4	53.5	43.3	56.4		NNE	Overcast—showery.
♃ 13	29.953	53.7	30.023	55.2	45	48.6	53.0	45.5	53.7	0.012	N	Overcast—light wind.
♀ 14	30.181	53.8	30.142	56.8	44	48.8	54.5	42.7	55.7		N	Fine—light clouds.
♄ 15	30.219	56.7	30.224	61.5	46	55.7	63.0	46.3	64.7		SSW	Fine—lightly cloudy.
☉ 16	30.329	63.7	30.292	64.2	45	60.0	68.3	45.7	69.2		SW	Fine and clear—light clouds.
☽ 17	30.240	64.2	30.175	66.2	53	64.6	73.0	52.6	75.0		WSW	Fine and clear.
♂ 18	30.058	67.3	29.969	68.8	48	67.3	73.4	56.3	74.7		SSW	Fine and clear—light clouds.
☿ 19	29.931	71.3	29.920	68.8	48	63.6	65.6	51.3	69.3		WNW	A.M. Fine and clear. P.M. Lightly overcast.
♃ 20	29.977	70.7	29.940	70.8	52	68.6	64.6	55.7	69.5		ESE	Fine and clear—light clouds.
♀ 21	29.834	65.7	29.821	67.0	50	62.0	62.8	51.7	65.4		E	Fine and clear—thunder and lightning at night.
● ♄ 22	29.810	61.5	29.914	66.8	51	57.8	64.3	50.2	65.2	0.361	SSW	A.M. Overcast. P.M. Fine and clear.
☉ 23	29.906	65.8	29.834	67.7	54	63.0	63.8	51.6	68.5		E	Clear—cloudy. At 4½ P.M. violent storm during 40 minutes.
☽ 24	29.680	64.7	29.740	68.4	60	64.0	68.8	55.7	68.3	0.489	ESE	Fine—cloudy—heavy rain at 6 A.M.
♂ 25	29.660	72.3	29.581	69.6	53	66.4	65.2	52.4	69.5		S	Overcast—light clouds.
☿ 26	29.445	64.8	29.449	66.2	51	59.7	59.8	53.3	63.9	0.347	W	A.M. Fine. P.M. Rain.
♃ 27	29.525	62.3	29.554	65.8	46	56.6	60.5	49.7	61.7	0.069	W var.	Fine—cloudy.
♀ 28	29.841	59.7	29.941	62.2	47	52.9	53.5	50.3	58.3		NNW	Cloudy and showery.
♄ 29	30.104	66.3	30.048	64.6	41	58.4	63.5	47.8	66.0	0.022	NW	Fine—lightly overcast.
☉ 30	29.803	60.6	29.769	63.6	52	56.0	59.7	52.3	60.3	0.144	SW	Broken clouds. Rain A.M.
☽ 31	29.901	66.0	29.978	64.2	49	59.4	61.8	48.6	63.3	0.111	WSW	Fine—cloudy.
	Mean	Mean	Mean	Mean	Mean	Mean.	Mean.	Mean	Mean	Sum		
	29.884	63.1	29.875	64.5	49.3	59.0	62.3	50.0	64.8	2.007		

Monthly Mean of the Barometer, corrected for Capillarity and reduced to 32° Fahr. ..... { 9 A.M. 29.808 } { 3 P.M. 29.796 }

OBSERVANDA.

Height of the Cistern of the Barometer above a Fixed Mark on Waterloo Bridge..... = 83 feet 2½ in.

..... above the mean level of the Sea (presumed about) ..... = 95 feet.

The External Thermometer is 2 feet higher than the Barometer Cistern.

Height of the Receiver of the Rain Gauge above the Court of Somerset House ..... = 79 feet 0 in.

The hours of observation are of Mean Time, the day beginning at Midnight.

The Thermometers are graduated by Fahrenheit's Scale.

The Barometer is divided into inches and decimals.

METEOROLOGICAL JOURNAL FOR JUNE, 1830.

1830. June.	9 o'clock, A.M. Barom.	9 o'clock, A.M. Attach. Therm.	3 o'clock, P.M. Barom.	3 o'clock, P.M. Attach. Therm.	Dew Point at 9 A.M. in degrees of Fahr.	External Thermometer. Fahrenheit. 9 A.M.	Fahrenheit. 3 P.M.	Self-registering. Lowest.	Self-registering. Highest.	Rain, in inches. Read off at 9 A.M.	Direction of the Wind at 9 A.M.	Remarks.
♂ 1	30.054	61.8	30.071	64.3	48	60.0	61.8	54.2	65.3		WSW	Fine—lightly cloudy.
☿ 2	30.161	63.7	30.113	65.8	49	59.7	67.2	49.3	69.3		SSW	Fine—light clouds.
♃ 3	29.807	64.6	29.612	62.8	53	61.6	60.5	53.5	62.3	0.125	SSE	Cloudy. Light rain with unsteady wind.
♀ 4	29.620	59.6	29.789	62.8	51	54.3	58.5	51.7	60.4	0.581	SW	Lowering—light brisk wind.
♄ 5	30.036	65.4	30.002	65.0	53	58.5	66.5	48.6	68.3		W	Lightly overcast.
○ ☉ 6	30.036	70.7	29.963	68.0	50	65.3	66.5	52.8	69.6		SW	Clear—cloudy.
☽ 7	29.908	63.7	29.900	67.2	55	59.5	60.5	54.5	64.2	0.108	SSW	Overcast—light rain.
♂ 8	30.045	62.5	30.065	64.3	50	55.7	58.9	52.3	59.7	0.019	N	Lightly overcast.
☿ 9	30.070	59.4	30.024	60.8	51	53.7	56.5	51.3	57.5		N	Overcast—Rain P.M.
♃ 10	30.006	57.7	29.986	60.2	52	52.6	56.8	49.7	59.8	0.042	NNW	Overcast—Light rain P.M.
♀ 11	30.002	66.7	29.977	65.8	53	59.8	63.4	49.8	65.7	0.033	W	Lightly overcast.
♄ 12	29.751	62.0	29.735	64.8	58	58.3	64.2	55.3	65.3	0.125	SSW	A.M. Light rain. P.M. Fine—light clouds.
☉ 13	29.784	61.7	29.739	64.2	46	58.4	58.8	47.7	64.3		W	Fine—showery. Thunder at noon.
☽ 14	29.688	59.3	29.637	63.2	46	56.3	58.0	51.6	62.7	0.125	NNW	Overcast—Light rain P.M.
♂ 15	29.688	63.7	29.708	61.0	46	55.0	56.5	48.3	59.7	0.278	N	Fine—light clouds and wind.
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♂ 29	29.898	72.6	29.939	72.5	57	68.6	71.4	58.8	73.2		W	Fine and clear—light clouds.
☿ 30	30.060	73.7	30.045	70.6	57	65.5	69.0	53.4	69.7		NNW	Fine and clear—cloudy.
	Mean	Mean	Mean	Mean	Mean	Mean	Mean	Mean	Mean	Sum		
	29.844	64.4	29.836	65.1	51.3	59.2	62.8	52.0	65.2	2.091		

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